Homework Sheet #14

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Homework Set #14: Monday, 20 December 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(46) Let $\mathbb{P}$ be a partial order in which every antichain is countable; commonly called a *ccc partial order*. Let G be an M-generic filter on $\mathbb{P}$.

a. Prove: if X and Y are in M and if $f : X \to Y$ is a function in $M[G]$ then there is a map $F : X \to [Y]^{\leqslant\aleph_0}$ in M such that $(\forall x \in X)(f(x) \in F(x))$.

b. Prove: for all ordinals α in M we have $\mathrm{cf}^{M[G]}\,\alpha = \mathrm{cf}^M\,\alpha$.

c. Let $f : \omega_1 \to \omega_1$ be a function in $M[G]$. Show that there is a closed and unbounded set C in M such that for all $\delta \in C$ we have $(\forall\alpha \in \delta)(f(\alpha) < \delta)$.

d. Use the previous part to show that if $S \in M$ is stationary in ω_1 *in* M then S is also stationary in ω_1 *in* $M[G]$. *Hint*: Show that a closed unbounded set $C \in M[G]$ contains a closed unbounded set D such that $D \in M$.

(47) Assume M satisfies GCH and let G be M-generic on $\mathrm{Fn}(\omega_\omega \times \omega, 2)$. Calculate the values of $2^{\aleph_0}$ and $2^{\aleph_\omega}$ in $M[G]$.

(48) Let M be an arbitrary countable model of ZFC. Let $\mathbb{P} = \mathrm{Fn}(\omega_1 \times \omega, 2, \aleph_1)^M$ and let G be M-generic on $\mathbb{P}$.

a. For a subset x of ω in M let D_x be the set of $p \in \mathbb{P}$ for which there is an $\alpha \in \omega_1$ such that $\{\alpha\} \times \omega \subseteq \mathrm{dom}\, p$ and $x = \{n : p(\alpha, n) = 1\}$. Verify that D_x is dense in $\mathbb{P}$.

b. For $\alpha \in \omega_1^M$ let $x_\alpha = \{n : \bigcup G(\alpha, n) = 1\}$. Prove that $\mathcal{P}^M(\omega) = \{x_\alpha : \alpha \in \omega_1^M\}$.

c. Prove that $\mathcal{P}^{M[G]}(\omega) = \mathcal{P}^M(\omega)$.

d. Deduce that CH holds in $M[G]$.