QUICK SOLUTIONS TO HOMEWORK 14

Exercise (46)

a. You can (almost) literally copy page 5 from the notes for 2021-12-13; there the statement is proved for $\mathbb{P} = \mathrm{Fn}(\omega_2 \times \omega, 2)$ but it only uses the property that antichains are countable.

b. The same as for $\mathrm{Fn}(\omega_2 \times \omega, 2)$ again, see page 6 of the notes mentioned above.

c. Given f take F as in **a**. Then the set of δ that satisfy $(\forall\alpha \in \delta)(F(\alpha \subseteq \delta)$ is closed unbounded.

d. Apply the hint, and let $f : \omega_1 \to D$ be the unique order-preserving bijection. The set C from part **c** is as required.

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The calculations given in class will show that in $M[G]$ we have $2^{\aleph_n} \leq \aleph_{\omega+1}$ for all n: this uses that $\aleph_\omega$ has $\aleph_\omega^{\aleph_0}$ countable subsets, and by the GCH that power is equal to $\aleph_{\omega+1}$.
Since we have $2^{\aleph_0} \geq \aleph_\omega$ and $2^{\aleph_0} \neq \aleph_\omega$, we find that $2^{\aleph_n} = \aleph_{\omega+1}$ for all n. Now cardinal arithmetic dictates that $2^{\aleph_\omega} = \aleph_{\omega+1}$.

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a. If $p \in \mathbb{P}$ then $\operatorname{dom} p$ is countable, so take α such that $\{\alpha\}\times$ is disjoint from $\operatorname{dom} p$, use that set to extend p to a q that is in D_x.

b. Part **a** shows that $\mathcal{P}^M(\omega)$ is a subset of $\{x_\alpha : \alpha \in \omega_1^M\}$. For the converse: for every α the set $E_\alpha = \{p : \{\alpha\} \times \omega \subseteq \operatorname{dom} p\}$ is dense (as in part **a**). Take $p \in G \cap E_\alpha$, then $x_\alpha = \{n : p(\alpha, n) = 1\}$ is defined from members of M, hence in M.

c. We saw in class that this partial order adds no new countable subsets of members of M.

d. We have seen in class that $\omega_1^{M[G]} = \omega_1^M$, together with **c** and **b** this shows that there is a surjection from $\omega_1^{M[G]}$ onto $\mathcal{P}^{M[G]}(\omega)$.