Homework Sheet #13

MasterMath: Set Theory
2021/22: 1st Semester
K. P. Hart, Steef Hegeman, Benedikt Löwe & Robert Paßmann

Deadline for Homework Set #13: Monday, 13 December 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(43) Let $\mathbb{P}$ be a partial order. Define, by recursion on α:
- $N_0 = \emptyset$,
- $N_{\alpha+1} = \mathcal{P}(N_\alpha \times \mathbb{P})$, and
- $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ if α is a limit ordinal.

Prove that $\bigcup_\alpha N_\alpha$ is equal to the class $V^{\mathbb{P}}$ of all $\mathbb{P}$-names.

(44) Let M be a countable model of ZFC and $\mathbb{P} \in M$ a partial order. Generalize Problem (41) from last week and prove the following general statement: a filter G on $\mathbb{P}$ is M-generic if and only if it interescts every maximal antichain in $\mathbb{P}$ that is an element of M.

(45) The results in class imply that $p \Vdash (\exists x)(\varphi(x,\tau))$ is equivalent to the set

$$E = \big\{q \leqslant p : (\exists\sigma)\big(q \Vdash \varphi(\sigma,\tau)\big)\big\}$$

being dense below p.
This problem proves that it is in fact equivalent to

$$(\exists\sigma)\big(p \Vdash \varphi(\sigma,\tau)\big).$$

(as one would probably expect).

a. Prove that there is a maximal antichain A in E.

b. Prove that there is a function that chooses for every $q \in A$ a name σ_q such that $q \Vdash \varphi(\sigma_q,\tau)$.

Let $D = \bigcup\{\operatorname{dom}\sigma_q : q \in A\}$. Define

$$\sigma = \{\langle\pi, r\rangle : (\exists q \in A)(\exists t \in \mathbb{P})(r \leqslant q \wedge r \leqslant t \wedge \langle\pi,t\rangle \in \sigma_q)\}$$

c. Prove that $p \Vdash \varphi(\sigma,\tau)$. *Hint*: If G is M-generic then $G \cap A$ consists of exactly one point q; prove that $\operatorname{val}(\sigma,G) = \operatorname{val}(\sigma_q,G)$.

<u>DEFINITION</u>

IN GENERAL: IF φ IS A FORMULA AND $\tau_1, \dots, \tau_k$ ARE NAMES THEN

$p \Vdash \varphi(\tau_1,\dots,\tau_k)$ IFF

FOR <u>ALL</u> M-GENERIC G WITH $p \in G$ WE HAVE
$M[G] \models \varphi(\mathrm{VAL}(\tau_1,G),\dots,\mathrm{VAL}(\tau_k,G))$

(43) Let $V^{\mathbb{P}}$ be the class of all $\mathbb{P}$-names and let $\bigcup_\alpha N_\alpha$ be defined as above

We further define the rank of a $\mathbb{P}$-name recursively:

A $\mathbb{P}$-name of rank α (α an ordinal) is a $\mathbb{P}$-name

$\tau = \{(\tau_i, p_i) \mid i < i_0\}$ where $p_i \in \mathbb{P}$ and τ_i is a $\mathbb{P}$-name of rank $< \alpha$

We now show that the $\mathbb{P}$-names of rank α are equivalent to N_α

$N_0 = \emptyset$ and the only $\mathbb{P}$-name of rank 0 is $\emptyset$

Hence N_0 is equivalent (in fact equal) to the $\mathbb{P}$-name of rank 0.

Assume for an ordinal α, they are equivalent

$$N_{\alpha+1} = \mathcal{P}(N_\alpha \times \mathbb{P}) \cong \mathcal{P}(\{(\tau, p) \mid \tau \text{ is a } \mathbb{P}\text{-name of rank } \alpha, p \in \mathbb{P}\})$$
$$= \{\{(\tau, p) \mid \tau \text{ is of rank} < \alpha, p \in \mathbb{P}\}\}$$

which is exactly all the $\mathbb{P}$-names of rank $\alpha+1$

In the limit case assume the equivalence holds for all $\beta < \gamma$

$$N_\gamma = \bigcup_{\beta<\gamma} N_\beta \cong \bigcup_{\beta<\gamma} \{\mathbb{P}\text{-names of rank } \beta\}$$
$$= \text{exactly the } \mathbb{P}\text{-names of rank } \gamma.$$

Hence they are equivalent.

(44) Suppose that G is M-generic. Let A be a maximal antichain of $\mathbb{P}$ that is in M.

Let $D = \{p \in \mathbb{P} \mid \exists a \in A \ (p \leq a)\}$

The maximality of A ensures D is dense, and it is open as a union of opens.

Thus $D \cap G \neq \emptyset$. $\Rightarrow \exists p \in D \cap G$.

$\Rightarrow \exists a \in A \quad p \leq a \Rightarrow a \in G$ by upward closure

Hence $a \in A \cap G$.

Conversely suppose that for every maximal antichain A that is in M, $A \cap G \neq \emptyset$.

Fix $D \in M$ a dense open subset of $\mathbb{P}$

Let $A \subseteq D$ be a maximal antichain of D.

Then by (40) A is maximal in $\mathbb{P}$ and furthermore $A \in M$.

$\Rightarrow A \cap G \neq \emptyset$

$A \cap G \subseteq D \cap G \Rightarrow D \cap G \neq \emptyset$.

(45) a. E is partially ordered as a subset of a partial order. By (40 a) every antichain in a partial order is contained in a maximal antichain. Thus proving that a maximal antichain exists reduces to proving that any antichain exists. However, if E is nonempty, then for any $q \in E$, $\{q\}$ is an antichain (if E is empty then the problem is trivial) so $\exists$ a maximal antichain A containing $\{q\}$.

b. Let $\mathcal{F}$ be a family of sets F_q where

$$\forall q \in A \qquad F_q = \{\sigma \mid q \Vdash \varphi(\sigma, \tau)\}$$

Then by the axiom of choice, there exists a choice function that selects $\sigma_q \in F_q$ for every $q \in A$.

c. We need to show that for all M-generic G with $p \in G$

$$M[G] \models \varphi(\mathrm{val}(\sigma, G), \mathrm{val}(\tau, G))$$

G is M-generic $\Rightarrow G \cap A \neq \emptyset$. Now suppose

$$n, q \in G \cap A, \; n \neq q$$

Then $\exists\, r \in G$ st $r \leq n$ and $r \leq q$

$\Rightarrow n \parallel q$. But $n, q \in A \Rightarrow n \perp q$. Contradiction

$\Rightarrow$ $G \cap A = \{q\}$ for some q.

$q \Vdash \varphi(\sigma_q, \tau)$

$\Rightarrow$ $M[G] \models \varphi(val(\sigma_q, G), val(\tau, G))$

Hence if $val(\sigma_q, G) = val(\sigma, G)$, then we are done.

$val(\sigma, G) = \{val(\mu, G) \mid \exists n \in G \ \langle \mu, n\rangle \in \sigma\}$

$val(\sigma_q, G) = \{val(\mu, G) \mid \exists n \in G \ \langle \mu, n\rangle \in \sigma_q\}$

We have defined

$$\sigma = \{\langle \pi, r\rangle \mid (\exists n \in A)(\exists t \in \mathbb{P}) \ r \leq n \wedge r \leq t \ \wedge \langle \pi, t\rangle \in \sigma_n\}$$

Suppose $\alpha \in val(\sigma_q, G)$. Then $\exists \pi$ such that $\alpha = val(\pi, G)$ and $\exists n \in G \ \langle \pi, n\rangle \in \sigma_q$

$\Rightarrow \langle \pi, n\rangle \in \sigma \Rightarrow \alpha \in val(\sigma, G)$

$\Rightarrow val(\sigma_q, G) \subseteq val(\sigma, G)$

Suppose $\alpha \in val(\sigma, G)$. Then $\exists \pi$ such that

$\alpha = val(\pi, G)$ and $\exists r \in G \ \langle \pi, r\rangle \in \sigma$

$\Rightarrow (\exists n \in A)(\exists t \in \mathbb{P}) \ r \leq n \wedge r \leq t \wedge \langle \pi, t\rangle \in \sigma_n$

But upwards closure $\Rightarrow n \in G \Rightarrow n \in G \cap A$

$\Rightarrow n = q$

$\Rightarrow \langle \pi, t\rangle \in \sigma_q \Rightarrow \alpha \in val(\sigma_q, G)$

$\Rightarrow val(\sigma, G) \subseteq val(\sigma_q, G)$

$\therefore$ $val(\sigma, G) = val(\sigma_q, G)$.