40. a) Every antichain is contained in a maximal antichain.

$\triangleright$ Suppose that A is an antichain.

Take $\mathbb{B} = \{ B \mid A \subseteq B,\ B \text{ is an antichain} \}$.

$\mathbb{B}$ is a partially ordered set with $\subseteq$.

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Let us apply Zorn's lemma to obtain a maximal element of $\mathbb{B}$.

First we show that $\forall C: C\text{-chain in } \mathbb{B} \Rightarrow$ C has an ~~maximal element~~. upper bound.

Let C be a chain in $\mathbb{B}$. Then take $\bigcup C$. It is an antichain and $\bigcup C \supseteq A$, since $\forall c \in C: c \supseteq A$. Thus $\bigcup C$ is an upper bound.

By Zorn's lemma: $\exists A': A'$ is max in $\mathbb{B}$.

A' is an antichain and $A \subseteq A'$. $\triangleleft$

b). If A is a max antichain in D-dense, then A is max antichain in $\mathbb{P}$.

$\triangleright$ Suppose A is not max in $\mathbb{P}$. Then $\exists p \in \mathbb{P}: A \cup \{p\}$ is an antichain.

Since D is dense, we have:

$\exists q: q \leq p \wedge q \in D$.

Thus, $A \cup \{q\}$ is also an antichain.

$\triangleright$ Suppose $A \cup \{q\}$ is not an antichain.

Then $\exists q' \in A \; \exists r \in \mathbb{P}$: $r \leq q$; $r \leq q'$

Then, $\exists r \in \mathbb{P}$: $r \leq q \leq p$; $r \leq q'$

Thus, $A \cup \{p\}$ is not an antichain $\triangleleft$.

It is a contradiction (since $q \in D$ and A is a max antichain in D). Therefore A is a max antichain in $\mathbb{P}$.

$\triangleleft$

41. G is M-generic filter on $\overset{\mathbb{P}}{Fn(\omega_2^M \times \omega, 2)}$ iff $\forall A \overset{\in M}{:}$ A-max antichain of $\mathbb{P}$, $A \cap G \neq \emptyset$.

($\Rightarrow$) Suppose G is a filter on $Fn(\omega_2^M \times \omega, 2)$ and $\exists A \overset{\in M}{:}$ A-max antichain of $\mathbb{P}$ and $A \cap G = \emptyset$.

We are going to build a dense set in M, s.t. $G \cap D = \emptyset$.

~~Since~~ First notice that: $\forall p \in G \; \exists q \in A$: p, q are compatible.

Recursively define sequence of $\langle D_i : i \in \varkappa \rangle$, where $\varkappa = |\mathbb{P}|$.

Put $D_0 = A$.

Suppose for all $\xi < \delta$ D_ξ is defined.

$f(\delta) = p$, M

Take $p \in P$ where f is a bijection between P and κ.

Then $p \notin G$ or $p \in G$. If $p \notin G$, take

$$D_\delta = \bigcup_{\xi<\delta} D_\xi \cup \{p\}$$

If $p \in G$, take $q \in A$, s.t. q and p are compatible. Then $\exists r \in P$: $r \leq p$; $r \leq q$.

Thus $r \notin G$, because o/w $q \in G$, since G is a filter. Thus take $D_\delta = \bigcup D_\xi \cup \{r\}$.

Take $D = \bigcup_{\alpha \in \kappa} D_\alpha$.

~~Suppose~~

Let us show that D is dence.

Suppose $p \in P$. Then there is $\alpha \in \kappa$, s.t. $f(\alpha) = p$ (where f is a ~~an isomorphism~~ bijection).

Take D_α. Then $D_\alpha = \bigcup_{\beta<\alpha} D_\beta \cup \{p\}$

or $D_\alpha = \bigcup_{\beta<\alpha} D_\beta \cup \{r\}$, where $r \leq p$

Thus, $\exists r: r \in D$ and $r \leq p$.

Hence, D is dense.

Also we have: $D \cap G = \emptyset$.

$\Rightarrow$ G is not M-generic.

$(\Leftarrow)$

Suppose G is a filter and G is not M-generic. Then $\exists D: D$ is dense in M for $\mathbb{P}$, and $G \cap D = \emptyset$.

Thus, take any antichain A in D.

Then it is contained in a maximal antichain A'.

By 40.b): A' is max in $\mathbb{P}$. Since $A' \subseteq D$, then $A' \cap G = \emptyset$. (also A' is in M, since D is in M).

$\square$

42 a) Construct an antichain in $Fn(X, 2, \aleph_1)$ that is of cardinality $2^{\aleph_0}$.

$\triangleright$ take any $Z \subseteq X$, Z is countable.

Take $\mathbb{B} = \{ \chi_A \mid A \subseteq Z, \chi_A$ is a characteristic function$\}$.

$|\mathbb{B}| = 2^{\aleph_0}$ and $\mathbb{B}$ is an antichain. $\square$

b). ▷ Since $A \subseteq [\kappa]^{\leq \aleph_0} = \kappa^{\aleph_0} = \kappa$

▷ By theorem • related to cardinal arithmetic we have that:

$$\mu^{\aleph_0} < \kappa \quad \forall \mu < \kappa, \ cf(\kappa) = \kappa$$
$$\Rightarrow \kappa^{\aleph_0} = \kappa$$

◁

Therefore we can built a sequence $\langle a_\alpha : \alpha \in \kappa \rangle$. Now the proof of relativisation of Δ-system theorem can be repeated.

◁

c). ▷ Suppose $|A| > 2^{\aleph_0}$. Take $|A'| = (2^{\aleph_0})^+$. $A' \subseteq A$. Since $\forall \lambda : \lambda < (2^{\aleph_0})^+ \Rightarrow \lambda^{\aleph_0} < \kappa$, we can apply 42 b). Then just repeat the proof of the last theorem of the lecture

◁.