Homework 11: Set Theory

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(37)

1. This is immediate from the definitions.
2. Suppose that φ is absolute for M, N. Let $m_1, \dots m_k \in M$, and suppose that $(\exists x \varphi(x, m_1, \dots, m_k))^M$ holds. Per definition, this means that $\exists x(x \in M \wedge \varphi(x, m_1, \dots, m_k)^M)$ holds. Now note that $x \in N$, since $M \subseteq N$ and by assumption $\varphi(x, m_1, \dots, m_k)^N$ holds. Thus, we have $(\exists x \varphi(x, m_1, \dots, m_k))^N$ as desired.
3. Suppose that φ is absolute for M, N. Let $m_1, \dots m_k \in M$, and suppose that $(\forall x \varphi(x, m_1, \dots, m_k))^N$ holds. Per definition, this means that $\forall x(x \in N \wedge \varphi(x, m_1, \dots, m_k)^N)$ holds. By assumption, we have $\varphi(x, m_1, \dots, m_k)^M$ holds if $\varphi(x, m_1, \dots, m_k)^N$ holds (for $x \in M$). Since $M \subseteq N$, it follows that $(\forall x \varphi(x, m_1, \dots, m_k))^M$ holds.

(38)

1. Let $\psi(A, X, R) := (R \text{ is a transitive relation on } X) \wedge \forall u \in X(\neg uRu) \wedge ((A \subseteq X \wedge \exists x \in A) \rightarrow \exists y \in A(\forall z \in A(y \neq z \rightarrow yRz)))$.

   Note here that subset of and transitivity are Δ_0 concepts (see lemma 12.10 of Jech).
2. I'm not going to write down this formula because it will be too long, but note that the following properties are absolute:
   - R is a partial order on X
   - f is a function
   - $x \in \text{dom}(f)$ iff $(x \subseteq X \wedge \exists y \in x)$.
   - $\forall x \in \text{dom}(f)(\exists y \in x(\forall z \in x(y \neq z \rightarrow yRz)))$.

   Putting these properties together, tells us that f is a function mapping a non-empty subset of X to a least element of X. Thus, they give a Δ_0 formula of the form that we want.
3. Suppose M and N satisfy the axioms of used in the proof of the representation theorem of well orders. Then, if (X, R) is a well-order, there is a unique ordinal α in M (and thus also in N) such that $(X, R) \cong (\alpha, \in)$. Now, the formula given in part (b) is upward absolute and the formula given in (a) is downward absolute.

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1. The order is clearly irreflexive. Suppose $(i, m)R(j, n)R(k, l)$. If $i = 0$ and $j = 1$, then $k = 1$, so $(i, m)R(k, l)$. If $i = j = 0$ and $m < n$, then $k = 1$, or $k = 0$ and $n < l$. In either case, $(i, m)R(k, l)$. If $i = j = 1$ and $m > n$, then $k = 1$ and $n > l$, so $(i, m)R(k, l)$. Now suppose that $(i, m), (j, n) \in X$. If $i \neq j$, then the elements are comparable. Suppose $i = j$. If $n = m$, then $(i, m) = (j, n)$, and if $n \neq m$, then the elements are again comparable. We conclude that R is a linear order.

   We claim that R is not a well order. Let $A := \{(i, n) \in X \mid i = 0\}$. Clearly, this set is non-empty does not have a least element.

2. The inclusion $V_\omega \subseteq M$ is clear. It is also not difficult to see that $(0, n) \in V_\omega$ for any $n \in \omega$, so $\{0\} \times \omega \subset V_\omega$. It follows that $\mathcal{P}(\{0\} \times \omega) \subseteq V_{\omega+1}$. In a similar manner, we see that $X, R \subset V_\omega$, so $\{X, R\} \subseteq V_{\omega+1}$. We conclude that $M \subseteq V_{\omega+1}$.

   We now check that M is transitive. Let $x \in M$. Suppose $x \in V_\omega$, then $x \subseteq M$ by the transitivity of V_ω. Suppose $x \in \mathcal{P}(\{0\} \times \omega)$, then $x \subseteq V_\omega \subseteq M$. In the same way, we saw that $X, R \subseteq V_\omega \subseteq M$. We conclude that M is transitive.

3. Let $A \subseteq X$ be non-empty and such that $A \in M$. Supppose $(0, n) \in A$ for some $n \in \omega$, then the set $\{(0, m) \in A \mid m \in \omega\}$ is non-empty and clearly the least m such that $(0, m) \in A$ is the least element of A. Suppose that $(0, n) \notin A$ for any $n \in \omega$. Note however that any subset of $\{(1, n) \mid n \in \omega\}$ that is in V_ω must have been added at some finite stage, and there is thus a largest m such that $(1, m)$ lies in this subset. It follows that A has a least element. We conclude that any subset of X that lies in M has an R-least element.

4. R is a well order of X according to M, but not according to $V_{\omega+1}$. It follows that "R is a well order of X" is not absolute for transitive sets.