Homework Sheet #10

MasterMath: Set Theory
2021/22: 1st Semester
K. P. Hart, Steef Hegeman, Benedikt Löwe & Robert Paßmann

Deadline for Homework Set #10: Monday, 22 November 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(34) Some (standard) applications of Ramsey's theorem.

a. Let $\langle L, <\rangle$ be an infinite linearly ordered set. Prove that L has an infinite subset X that is well-ordered by $<$ or an infinite subset Y that is well-ordered by $>$.

b. Prove that every bounded sequence of real numbers has a convergent subsequence (the Bolzano-Weierstraß theorem). *Hint*: Find a monotone subsequence.

c. Let $\langle P, <\rangle$ be an infinite partially ordered set. Prove that P has an infinite subset C that is linearly ordered by $<$ (a chain) or an infinite subset U that is unordered by $<$, which means that if x and y in U are distinct then neither $x < y$ nor $y < x$.

(35) Another application of Ramsey's theorem. Here are four well-behaved families of subsets of ω:

(1) $\mathcal{A} = \{\{n\} : n \in \omega\}$,

(2) $\mathcal{B} = \{n : n \in \omega\}$,

(3) $\mathcal{C} = \{\omega \setminus \{n\} : n \in \omega\}$, and

(4) $\mathcal{D} = \{\omega \setminus n : n \in \omega\}$.

Let X be an infinite set and $\mathcal{S}$ an infinite family of subsets of X. Prove that there is a sequence $\langle x_n : n \in \omega\rangle$ of points in X and there is a sequence $\langle S_n : n \in \omega\rangle$ of members of $\mathcal{S}$ such that

$$\big\{\{m \in \omega : x_m \in S_n\} : n \in \omega\big\}$$

is equal to one of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. (So every infinite family of sets is well-behaved somewhere.)

a. Construct a sequence $\langle x_n : n \in \omega\rangle$ of points in X and a sequence $\langle \mathcal{S}_n : n \in \omega\rangle$ of infinite subfamilies of $\mathcal{S}$ such that $\mathcal{S}_0 = \mathcal{S}$ and for every n the following hold: either $\mathcal{S}_{n+1} = \{S \in \mathcal{S}_n : x_n \in S\}$ or $\mathcal{S}_{n+1} = \{S \in \mathcal{S}_n : x_n \notin S\}$, and in addition $\mathcal{S}_{n+1}$ is a proper subset of $\mathcal{S}_n$.

b. Choose $S_n \in \mathcal{S}_n \setminus \mathcal{S}_{n+1}$ for every n. Verify that if $x_m \in S_m$ then $x_m \notin S_n$ for all $n > m$ and, conversely, if $x_m \notin S_m$ then $x_m \in S_n$ whenever $n > m$.

c. Now consider the colouring $F : [\omega]^2 \to 4$ given by: if $i < j$ then

$$F\big(\{i,j\}\big) = \begin{cases} 0 & \text{if } x_i \notin S_j \text{ and } x_j \notin S_i \\ 1 & \text{if } x_i \notin S_j \text{ and } x_j \in S_i \\ 2 & \text{if } x_i \in S_j \text{ and } x_j \notin S_i \\ 3 & \text{if } x_i \in S_j \text{ and } x_j \in S_i \end{cases}$$

(36) The following example shows that with infinitely many colours one cannot even expect three-point homogeneous sets: $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$.
Enumerate $\mathbb{Q}$ as $\langle q_n : n < \omega\rangle$ and define $T : [\mathbb{R}]^2 \to \omega$ by

$$T\big(\{x,y\}\big) = \min\{n : q_n \text{ is \textit{strictly} between } x \text{ and } y\}.$$

Show that there are no three-point homogeneous sets for T.