34. a) Let $\langle L, < \rangle$ be an infinite linearly-ordered set.

Choose an element $x_0 \in L$.

Either the initial segment or the final segment at x_0 must be infinite

If the initial segment is infinite, we may consider it as a final segment of the total order $>$.

From now on we consider the final segment

$$X_0 = \{x \in L \mid x_0 < x\}$$

Pick $x_1 \in X_0$ and consider the final segment

$$X_1 = \{x \in L \mid x_1 < x\}$$

Define x_n and X_n recursively:

$$X_n = \{x \in L \mid x_n < x\}$$

Choose x_{n+1} from X_n

Then the set $X := \{x_n \mid n \in \omega\}$ is an infinite subset of L that is well-ordered (as it corresponds to the integers with the usual ordering).

In the other case, define Y in the same way, reversing the inequality at every step.

b) Let us take a bounded sequence of real numbers

$$\langle x_n \mid n \in \omega \rangle$$

Then take the complete graph on the integers. Color the edge (i,j) blue if $i<j$ and $x_i < x_j$ and red otherwise

By Ramsey there exists a complete infinite subgraph in which every edge is blue or every edge is red.

If the subgraph is blue, then the subsequence given by the vertices is increasing. If it is red, the subsequence is decreasing.

In either case, we have a bounded and monotonic - hence convergent - subsequence.

c) Let $\langle P, \leq \rangle$ be an infinite partially ordered set. Let P_0 be a countably infinite sub-poset of P.

We go similarly to the previous problem:

Let $c : [P_0]^2 \to 2$ be a map (essentially a coloring of a graph) defined by $c(\{x,y\}) = \begin{cases} 0 & x \nleq y \wedge y \nleq x \\ 1 & x \leq y \vee y \leq x \end{cases}$

Then by Ramsey $\exists$ C an infinite subset of P_0 such that $\forall$ $X \in [C]^2$, $c(X) = 0$ or $c(X) = 1$

If $c(X) = 0$ then C is an antichain (unordered)

If $c(X) = 1$ then C is a linear order

35. a) Let X be an infinite set and $\mathcal{S}$ an infinite family of subsets. We can pick some element x_0 of X.

Then let $\mathcal{S}_{01} = \{ S \in \mathcal{S} \mid x_0 \in S \}$ and $\mathcal{S}_{02} = \{ S \in \mathcal{S} \mid x_0 \notin S \}$

Then $\mathcal{S} = \mathcal{S}_{01} \cup \mathcal{S}_{02}$ and $\mathcal{S}$ is infinite, so either $\mathcal{S}_{01}$ or $\mathcal{S}_{02}$ must be infinite. Set $\mathcal{S}_0 = \mathcal{S}_{01}$ if $\mathcal{S}_{01}$ is infinite and $\mathcal{S}_0 = \mathcal{S}_{02}$ if $\mathcal{S}_{01}$ is not infinite.

Then pick $x_1 \in X \setminus \{x_0\}$ so that $\{ S \in \mathcal{S}_0 \mid x_1 \in S \} \neq \mathcal{S}_0$ and $\{ S \in \mathcal{S}_0 \mid x_1 \notin S \} \neq \mathcal{S}_0$

Recursively define $\mathcal{S}_{n+1}$ in the same way, by

$$\mathcal{S}_{n+1} = \{ S \in \mathcal{S}_n \mid x_n \in S \}$$

or

$$\mathcal{S}_{n+1} = \{ S \in \mathcal{S}_n \mid x_n \notin S \}$$

depending on which is infinite, and then pick x_{n+1} so that $\mathcal{S}_{n+2} \neq \mathcal{S}_{n+1}$. Then $\mathcal{S}_{n+1} \subsetneq \mathcal{S}_n$ $\forall$ $n \in \omega$.

b) Choose $S_n \in \mathcal{S}_n \setminus \mathcal{S}_{n+1}$ for every n.

Suppose that $x_m \in S_m$.

We also know that $S_m \notin \mathcal{S}_{m+1}$

$\Rightarrow$ $\mathcal{S}_{m+1} = \{S \in \mathcal{S}_m \mid x_m \notin S\}$. Otherwise we would have $S_m \in \mathcal{S}_{m+1}$

Now $\forall\, n > m$ $\mathcal{S}_n \subseteq \mathcal{S}_{m+1} \Rightarrow S_n \in \mathcal{S}_{m+1}$

$$\Rightarrow x_m \notin S_n$$

Conversely, if $x_m \notin S_m$, then we get

$\mathcal{S}_{m+1} = \{S \in \mathcal{S}_m \mid x_m \in S\}$.

Then $\forall\, n > m$ $\mathcal{S}_n \subseteq \mathcal{S}_{m+1} \Rightarrow S_n \in \mathcal{S}_{m+1}$

$$\Rightarrow x_m \in S_n.$$

c) Consider the coloring $F : [\omega]^2 \to 4$ given by: if $i < j$

$$F(\{i,j\}) = \begin{cases} 0 & x_i \notin S_j \wedge x_j \notin S_i \\ 1 & x_i \notin S_j \wedge x_j \in S_i \\ 2 & x_i \in S_j \wedge x_j \notin S_i \\ 3 & x_i \in S_j \wedge x_j \in S_i \end{cases}$$

Then by Ramsey $\exists\, H \subseteq \omega$ such that F is constant on $[H]^2$ and H is countably infinite.

We now consider $M = \{\{m \in \omega \mid x_m \in S_n\} \mid n \in \omega\}$

Suppose that

image of $[H]^2$

① $F[[H]^2] = \{0\}$: Then $x_m \in S_n \Leftrightarrow m \not< n$ and $n \not< m$
$\Rightarrow m = n$

$\Rightarrow M = \{\{n\} \mid n \in \omega\} = \mathcal{A}$

② $F[[H]^2] = \{1\}$: Then $x_m \in S_n \Leftrightarrow m \not< n \Leftrightarrow m \geq n$

$\Rightarrow M = \{\omega \setminus n \mid n \in \omega\} = \mathcal{D}$

③ $F[[H]^2] = \{2\}$: Then $x_m \in S_n \Leftrightarrow m < n$

$\Rightarrow M = \{\{j \in \omega \mid j < n\} \mid n \in \omega\} = \{n \mid n \in \omega\} = \mathcal{B}$

④ $F[[H]^2] = \{3\}$: Then $x_m \in S_n \Leftrightarrow m < n \vee n < m \Leftrightarrow m \neq n$
$\Rightarrow M = \{\omega \setminus \{n\} \mid n \in \omega\} = \mathcal{C}$.

36. Enumerate $\mathbb{Q}$ as $\langle q_n \mid n < \omega \rangle$ and define $T : [\mathbb{R}]^2 \to \omega$ by

$$T(\{x,y\}) = \min\{n \mid q_n \text{ is } \underline{\text{strictly}} \text{ between } x \text{ and } y\}$$

Let $\{x,y,z\} \subset \mathbb{R}$. WLOG we may say $x < y < z$

Then $T(\{x,y\}) = n$ where $x < q_n < y$

and $T(\{y,z\}) = m$ where $y < q_m < z$

$\Rightarrow q_n < q_m \Rightarrow n \neq m$. Hence $\{x,y,z\}$ cannot be a homogeneous set.