H1, (1)

We will prove that $(\mathbb{N}, E)$ satisfies the axioms of extensionality and union, but that it does not satisfy the axioms of pairing and separation.

Extensionality: Suppose $(\mathbb{N}, E)$ does not satisfy the axiom of extensionality for reductio, then without loss of generality we have $n < m$ (in $\mathbb{N}$) such that $\forall x(x \in n \leftrightarrow x \in m)$. However, $n \in m$ and $n \notin n$ by definition of E, this is a contradiction.

Pairing: Consider $4, 6 \in \mathbb{N}$. Suppose that the axiom of pairing holds for reductio, then there is an $n \in \mathbb{N}$ such that $x \in n$ if, and only if, $x = 4$ or $x = 6$. It must be the case that $n > 6$, but this means that $5 \in n$ as well. This is a contradiction since only 4 and 6 are contained in n.

Union: Let $n \in \mathbb{N}\backslash\{0, 1\}$ be arbitrary. To prove that the axiom of union holds we need to show that there is a set containing the elements of the elements of n. The elements of n are $0, 1, \ldots, n-1$, and the elements of all these elements consist of $0, 1, \ldots, n-2$. By definition of E, we only have $m \in n-1$ whenever $m \in 0, \ldots, n-2$. Hence there is a set precisely containing the elements of the elements of n. As $n \in \mathbb{N}\backslash\{0, 1\}$ was chosen arbitrarily, we now know that the desired property holds for elements in that set.

This leaves us with 0 and 1. As 0 does not have any elements, there also do not exist any elements of its elements. Because 0 does not contain any elements, all the elements of the elements of 0 are contained in 0 itself, as desired. For 1 the case is similar: only 0 is an element of 1, therefore there also are no elements of the elements of 1. So 0 is the union of 1.

So the union of every $n \in \mathbb{N}$ exists.

Separation: We provide a counterexample. Consider the formula $\varphi(u, p_1, p_2) = p_1 \in u \wedge u \in p_2$ and the corresponding instance of the separation scheme: $\forall x \forall p_1 \forall p_2 \exists y \forall u(u \in y \leftrightarrow ((u \in x) \wedge (p_1 \in u) \wedge (u \in p_2)))$. Let $x = 6$, $p_1 = 2$ and $p_2 = 5$. If the axiom scheme of separation were to hold, there must be an $n \in \mathbb{N}$ such that $\forall u(u \in n \leftrightarrow ((u \in 6) \wedge (2 \in u) \wedge (u \in 5)))$. However, if we consider the right side of the biconditional then we see that all $u \in n$ must be such that $u < 6$, $2 < u$ and $u < 5$. So $u = 3$ or $u = 4$. Now, there cannot be a set containing precisely 3 and 4, because then it must also contain 0, 1 and 2 by definition of E. We conclude that $\forall x \forall p_1 \forall p_2 \exists y \forall u(u \in y \leftrightarrow ((u \in x) \wedge (p_1 \in u) \wedge (u \in p_2)))$ is not true in $(\mathbb{N}, E)$, so the axiom scheme of separation fails.

H1, (2)

EXERCISE 2. In a directed graph given two nodes x, y we say that x is a predecessor of y in case that xRy, where R is the relation of the graph. Then the predecessors of x are all the nodes y such that yRx.

- (EXT). In a graph model, the axiom of extensionality says that if two nodes have the same predecessors (set of predecessors) then they are equal. Since in the graph there aren't two nodes with the same set of predecessors the axiom holds in the model.
- (PAIR). In a graph model, the axiom of pairing says that given nodes x, y (maybe $x = y$) there exists another node z whose predecessors are exactly x and y. Since the top node (let's call it x) is not the predecessor of any node, so there is no node such that its only predecessors are x, i.e. the axiom of pairing doesn't hold.
- (UNION). In a graph model, the axiom of union says that given a node X there exists a node Y such that the predecessors of Y are exactly the predecessors of the predecessors of X. This doesn't hold in the model, since if we take X as the top node the predecessors of the predecessors of the top node are all the other three nodes, but there is nodes, but there is no node with exactly those 3 nodes as predecessors.
- (POWER). In a graph model, $x \subseteq y$ means that the set of predecessors of x is a subset of the set of predecessors of y. So the axiom of powerset means that for any node X there is a node Y such that the predecessors of Y are exactly the nodes such that their predecessors are contained in the predecessors of X. It is clear that the only subnode of the bottom node is itself. Since there isn't a node whose only predecessor is the bottom node the axiom is false.
- (SEP). We take the formula $x = p_1$, then the axiom of separation for that formula says

$$\forall p_1, y \exists z \forall x . x \in z \leftrightarrow x \in y \land x = p_1 .$$

  But if we take p_1 as the bottom node and y as the left node then the formula would say that there is a node whose only predecessor is the bottom node. Since this is false we know that the axiom scheme of speration doesn't hold in the model.

Exercise 3

Let the finite directed graph $G = (V, E)$ be defined as followed: the set of vertices V contains only the vertex a. The edge relation E is defined as aEa. Then it is clear that the axiom of extensionality holds in G, since there is only one vertex, so the axiom becomes vacuously true. Also the axiom of pairing holds, since we have that the vertex a contains itself. Hence for any two vertices, which can both only be vertex a, there exists a vertex, namely a, that contains the pair a, a. The axiom of union also holds, since we have aEa the vertex a also contains the union of its own elements, namely a.

Then to see that the separation schema fails to holds in G, we consider the formula $\phi(z)$ to be $z \neq z$. This formula is a contradiction and would therefore only be true in a vertex that is the empty set. However, we do not have such a vertex here, since we only have a. Therefore we have found a counterexample.

H1, (3)

H1, (4)

Exercise 4. Consider the following *axiom of binary union*:

$$\forall x \forall y \exists u \forall z (z \in u \leftrightarrow (z \in x \vee z \in y)).$$

Show that every graph that satisfies the axioms of pairing and union also satisfies the axiom of binary union.

Solution. First, we assume the axioms of pairing and union:

$$\forall a \forall b \exists c \forall x (x \in c \leftrightarrow x = a \vee x = b)$$
$$\forall X \exists Y \forall u (u \in Y \leftrightarrow \exists z (z \in X \wedge u \in z).$$

Then, we choose arbirary sets (nodes of the graph) x and y. We will apply the axiom of pairing to obtain the set u' satisfying the property $z \in u' \leftrightarrow z = x \vee z = y$. We then apply the axiom of union to u' to obtain the set u such that for any z, $z \in u \leftrightarrow \exists z'(z' \in u' \wedge z \in z')$. But we know that the only elements of u' are x and y, so we obtain that $z \in u \leftrightarrow z \in x \vee z \in y$. $\square$