Group Interaction #9

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one group interaction of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

For this group interaction κ will always denote an infinite cardinal.
The purpose of this session is to prove a consequence of the Erdős-Dushnik-Miller theorem (Theorem 9.7 in Jech's book):

> If $\prec$ is some well-order of κ then there is a subset H of κ of cardinality κ on which $\in$ and $\prec$ coincide, which means that for all $\alpha, \beta \in H$ we have $\alpha \prec \beta$ iff $\alpha \in \beta$.

(1) Try to create alternative well-orders of some specific κ, say $\kappa = \omega_1$ or $\kappa = \omega_0$, and try to avoid large sets where the orders coincide.

(2) Let $\lhd$ be the lexicographic order of $\kappa \times \kappa$: $\langle \alpha, \beta \rangle \lhd \langle \gamma, \delta \rangle$ if $\beta \in \delta$ or $\beta = \delta$ and $\alpha \in \gamma$.
Let $\prec$ be the canonical well-order of $\kappa \times \kappa$ defined in Chapter 3 of Jech's book.
Find some large sets on which $\lhd$ and $\prec$ coincide.

For the rest of the session $\prec$ is some well-order of κ.
For every $\alpha \in \kappa$ we define two sets:

$$B(\alpha) = \{\beta \in \kappa : (\beta \in \alpha \wedge \beta \prec \alpha) \vee (\alpha \in \beta \wedge \alpha \prec \beta)\}$$

and

$$R(\alpha) = \{\beta \in \kappa : (\beta \in \alpha \wedge \alpha \prec \beta) \vee (\alpha \in \beta \wedge \beta \prec \alpha)\}$$

(3) Verify: if $|B(\alpha)| = \kappa$ then also $|B^+(\alpha)| = \kappa$, where $B^+(\alpha) = \{\beta \in \kappa : \alpha \in \beta \wedge \alpha \prec \beta\}$.

(4) Why is it desirable to have a sequence $\langle \alpha_\xi : \xi < \kappa \rangle$ such that $\alpha_\xi \in B^+(\alpha_\eta)$ whenever $\eta < \xi$?

(5) Assume κ is regular *and* that $|R(\alpha)| < \kappa$ for all α. Show that $\bigcap_{\alpha \in A} B^+(\alpha)$ has cardinality κ if $|A| < \kappa$.

(6) Assume κ is regular *and* that $|R(\alpha)| < \kappa$ for all α. Construct a sequence as in exercise (4).

(7) Verify that the above construction in the previous exercise also works inside some subset S of κ of cardinality κ with the property that $|R(\alpha) \cap S| < \kappa$ for all $\alpha \in S$.

(8) Assume a set S as in the previous exercise does not exist. That means: in every subset S of κ of cardinality κ there is an element α such that $|R(\alpha) \cap S| = \kappa$.
Construct a sequence $\langle \alpha_n : n \in \omega \rangle$ such that $\alpha_n \in R(\alpha_m)$ whenever $m < n$.

(9) Show that the existence of a sequence as in the previous exercise leads to a contradiction.

From now on we assume that a set as in Exercise 7 exists; for convenience we assume $S = \kappa$.

(10) Verify that we can indeed assume that $S = \kappa$.

(11) Verify that the result holds in case κ is regular.

From now on we assume that κ is singular and we fix, as in the proof of the Free Set Lemma, an increasing sequence $\langle \kappa_\xi : \xi < \operatorname{cf} \kappa \rangle$ of regular cardinals that is cofinal in κ, and such that $\operatorname{cf} \kappa < \kappa_0$.
We also set $I_0 = \kappa_0$ and $I_\xi = \kappa_\xi \setminus \bigcup_{\eta < \xi} \kappa_\eta$.

(12) Let $\xi < \operatorname{cf} \kappa$. Show that there are a subset J_ξ of I_ξ and an ordinal $\gamma_\xi < \operatorname{cf} \kappa$ such that $|J_\xi| = \kappa_\xi$ and $|R(\alpha)| < \kappa_{\gamma_\xi}$ for $\alpha \in J_\xi$.

(13) Show that there is a cofinal subset A of $\operatorname{cf}\kappa$ such that if $\eta, \xi \in A$ and $\eta < \xi$ then $\gamma_\eta < \xi$. *Hint*: if $\operatorname{cf}\kappa$ is uncountable then A can be cub; otherwise you have to construct $A \subseteq \omega$ recursively.

After renumbering we simply assume that $A = \operatorname{cf}\kappa$, that is, we assume that $\eta < \xi$ implies $\gamma_\eta < \xi$.

(14) Construct a sequence $\langle K_\xi : \xi < \operatorname{cf}\kappa \rangle$ such that $K_\xi \subseteq J_\xi$, $|K_\xi| = \kappa_\xi$, and if $\eta < \xi$ then

$$K_\xi \cap \bigcup \{R(\alpha) : \alpha \in K_\eta\} = \emptyset.$$

Hint: try $K_\xi = J_\xi \setminus \bigcup\{R(\alpha) : \alpha \in \bigcup_{\eta<\xi} J_\eta\}$.

(15) Apply the regular case for each ξ to find $L_\xi \subseteq K_\xi$ of cardinality κ_ξ such that $\prec$ and $\in$ agree on L_ξ. Prove that that $\prec$ and $\in$ agree on $\bigcup_{\xi<\operatorname{cf}\kappa} L_\xi$.