Group Interaction #7

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one group interaction of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

In class we saw the statement

> An infinite cardinal κ is singular if and only if there exist a cardinal $\lambda < \kappa$ and a family $\{S_\alpha : \alpha < \lambda\}$ of subsets of κ such that $|S_\alpha| < \kappa$ for all α and $\kappa = \bigcup_{\alpha<\lambda} S_\alpha$.
>
> The smallest such cardinal λ is equal to $\operatorname{cf} \kappa$.

(1) Prove the equivalence. Check whether your proof uses the Axiom of Choice. In fact, neither of the implications needs AC, so try to come up with a proof in ZF.

(2) Consider variations of the statement. Can the S_α be taken disjoint? Can the cardinalities of the S_α be all singular? All regular?

The Singular Cardinals Hypothesis (SCH) was introduced in class:

> For every singular cardinal κ: if $2^{\operatorname{cf}\kappa} < \kappa$ then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.

(3) Prove that the Generalized Continuum Hypothesis implies SCH.

(4) Theorem 5.22 in Jech describes the behaviour of the continuum function and cardinal exponentiation under the assumption of SCH.

> **Theorem 5.22.** *Assume that* SCH *holds.*
>
> (i) *If κ is a singular cardinal then*
> (a) $2^\kappa = 2^{<\kappa}$ *if the continuum function is eventually constant below κ,*
> (b) $2^\kappa = (2^{<\kappa})^+$ *otherwise.*
>
> (ii) *If κ and λ are infinite cardinals, then:*
> (a) *If $\kappa \le 2^\lambda$ then $\kappa^\lambda = 2^\lambda$.*
> (b) *If $2^\lambda < \kappa$ and $\lambda < \operatorname{cf}\kappa$ then $\kappa^\lambda = \kappa$.*
> (c) *If $2^\lambda < \kappa$ and $\operatorname{cf}\kappa \le \lambda$ then $\kappa^\lambda = \kappa^+$.*

It is prefaced by the statement that cardinal exponentiation is determined by the continuum function on regular cardinals. Describe the/an algorithm that allows you to calculate κ^λ for arbitrary κ and λ under the assumption that you have an oracle that tells you the value of 2^μ for *regular* μ, that oracle could be in the form of a (class) function F from the class of ordinals to itself such that $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ whenever $\aleph_\alpha$ is regular.

(5) Use your algorithm to determine the following powers, under the assumption that SCH holds and that $2^{\aleph_{\alpha+1}} = \aleph_{\omega+\alpha^2+1}$ for all α.

a. $\aleph_\omega^{\aleph_\omega}$

b. $\aleph_{\omega+2}^{\aleph_2}$

c. $\aleph_{\omega_1\cdot 2}^{\aleph_1}$

d. $\aleph_5^{\aleph_{\omega_1}}$