Group Interaction #6

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one *group interaction* of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

(1) Let λ be a limit ordinal. Remember the following definitions from Lecture VII and get a feeling for the notions by thinking of a few examples and non-examples:

(a) A set $C \subseteq \lambda$ is called *cofinal* if for all $\alpha \in \lambda$ there is some $\gamma \in C$ such that $\alpha \in \gamma$.

(b) A function $f : \alpha \to \lambda$ is called *cofinal* if its range is cofinal.

(2) If λ is a limit ordinal and κ is a cardinal, prove that the following statements are equivalent:

(i) There is a cofinal function $f : \kappa \to \lambda$.

(ii) There is a cofinal subset of λ of size at most κ.

(3) Note that, in general, statement (i) from (2) is *not* equivalent to

(iii) There is a cofinal increasing function $f : \kappa \to \lambda$ (i.e., if $\alpha < \beta$, then $f(\alpha) < f(\beta)$).

[*Hint.* Any surjection is cofinal.]

(4) Remember the definition of $\mathrm{cf}(\lambda)$ from Lecture VII: we defined $\mathrm{cf}(\lambda) := \min\{|C|\,; C$ is cofinal in $\lambda\}$. Show that if $\kappa = \mathrm{cf}(\lambda)$, then (i) from (2) and (iii) from (3) are equivalent.

[*Hint.* If $f : \kappa \to \lambda$ is cofinal, construct the increasing enumeration $\widehat{f}$ by recursion via $\widehat{f}(\alpha) := \min\{\beta \in \mathrm{ran}(f)\,;$ for all $\gamma \in \mathrm{ran}(f{\restriction}\alpha)$, we have $\gamma < \beta\}$. Show that $\widehat{f}$ is increasing and cofinal. Where did you use the fact that κ was of minimal size?]

(5) Assume that $f : \kappa \to \lambda$ is cofinal and increasing and that C is cofinal in κ. Check that $f[C]$ is cofinal in λ.

(6) Observe that (4) implies that that $\mathrm{cf}(\lambda) := \min\{\kappa\,;$ there is a cofinal increasing function from κ to $\lambda\}$.

(7) Use (5) and (6) to prove the claim made in Lecture VII that $\mathrm{cf}(\lambda)$ is always a regular cardinal.

(8) Deduce from (7) that κ is regular if and only if $\mathrm{cf}(\kappa) = \kappa$.

(9) Go through the argument from Lecture VII that constructs the least aleph fixed point by recursively defining $\alpha_0 := \aleph_0$, $\alpha_{n+1} := \aleph_{\alpha_n}$, and $\alpha := \bigcup\{\alpha_n\,; n \in \omega\}$. Check that you understand the claims made in the lecture: that α is an aleph fixed point and that $\mathrm{cf}(\alpha) = \aleph_0$.

(10) Let κ be any regular cardinal. Let us construct an aleph fixed point α with $\mathrm{cf}(\alpha) = \kappa$ by recursion on κ:

$$\begin{aligned} \alpha_0 &:= \aleph_0, \\ \alpha_{\gamma+1} &:= \aleph_{\alpha_\gamma}, \\ \alpha_\lambda &:= \bigcup\{\alpha_\gamma\,;\,\gamma < \lambda\} + 1 \text{ (for limit ordinals } \lambda). \end{aligned}$$

Prove that for all $\gamma < \kappa$, α_γ is not an aleph fixed point.

[*Remark.* Note that α_λ is not even a cardinal for limit ordinals λ.]

(11) Now define $\alpha := \bigcup\{\alpha_\gamma\,;\,\gamma < \kappa\}$ and show that α is an aleph fixed point.

(12) Clearly $\mathrm{cf}(\alpha) \leq \kappa$ (why?). Show that $\mathrm{cf}(\alpha) = \kappa$.

[*Remark.* This uses the assumption from (10) that κ is regular. Make sure that you understand where in your proof you used that assumption!]

(13) Let us elaborate on the remark in (12): Is it conceivable that there is a construction that achieves the same result for singular cardinals κ?

[*Remark.* This is a trick question: consider (7).]

(14) What can you say about the relationship between α and κ? Is it conceivable that $\alpha = \kappa$? What properties would κ need to have in order for this to happen?