Group Interaction #5

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one *group interaction* of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

In the fifth group interaction, we prove Hessenberg's theorem.

(1) Read the following excerpt from Jech's book (pp. 30–31):

> **The Canonical Well-Ordering of $\alpha \times \alpha$**
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> We define a well-ordering of the class *Ord* $\times$ *Ord* of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the *canonical well-ordering* of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class *Ord*, and we have a one-to-one function Γ of Ord^2 onto *Ord*. For many α's the order-type of $\alpha \times \alpha$ is α; in particular for those α that are alephs.
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> We define:
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> (3.12) $$(\alpha,\beta) < (\gamma,\delta) \leftrightarrow \begin{array}{l} \text{either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\}, \\ \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma, \\ \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\},\ \alpha = \gamma \text{ and } \beta < \delta. \end{array}$$
>
> The relation $<$ defined in (3.12) is a linear ordering of the class *Ord* $\times$ *Ord*. Moreover, if $X \subset Ord \times Ord$ is nonempty, then X has a least element. Also, for each α, $\alpha \times \alpha$ is the initial segment given by $(0,\alpha)$. If we let
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> $$\Gamma(\alpha,\beta) = \text{the order-type of the set } \{(\xi,\eta) : (\xi,\eta) < (\alpha,\beta)\},$$
>
> then Γ is a one-to-one mapping of Ord^2 onto *Ord*, and
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> (3.13) $$(\alpha,\beta) < (\gamma,\delta) \quad \text{if and only if} \quad \Gamma(\alpha,\beta) < \Gamma(\gamma,\delta).$$

(2) Check Jech's claim in this text that $<$ is a linear order.

(3) Calculate the values of $\Gamma(i,j)$ for $i,j \leq 3$ (or, if you like, $i,j \leq 4$).

(4) Draw a picture of what the function Γ is doing if you restrict it to $\omega \times \omega$.

(5) Check that $(0,\omega)$ is the $<$-least pair bigger than all pairs in $\omega \times \omega$. What is $\Gamma(0,\omega)$?

(6) For $n \in \omega$, determine the values of $\Gamma(n,\omega)$, $\Gamma(\omega,n)$, $\Gamma(\omega,\omega)$, and $\Gamma(\omega,\omega+1)$.

(7) Check Jech's claim in this text that if X is a nonempty class of pairs of ordinals, then it has a $<$-least element.

(8) Check Jech's claim in this text that for each α, $\alpha \times \alpha$ is the initial segment of $<$ given by $(0,\alpha)$.

(9) Check Jech's claim that Γ is a one-to-one (injective) mapping.

(10) Hessenberg's Theorem says: for any infinite ordinal α, there is a bijection between α and $\alpha \times \alpha$. Convince yourself that it is enough to show the claim for *initial ordinals*, i.e., Alephs.

(11) Now prove Hessenberg's theorem by showing it for all $\aleph_\gamma$ by induction on γ.

[*Hint.* Use the mapping Γ.]

(12) Use Hessenberg's theorem to show that for infinite cardinal numbers $\kappa \leq \lambda$, the following three sets are in bijection with each other: λ, $\kappa \times \lambda$, and the disjoint union of κ and λ (i.e., $\kappa \times \{0\} \cup \lambda \times \{1\}$).