Group Interaction #4

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one *group interaction* of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

In the fourth group interaction, we understand the von Neumann levels as models of set theory. In the lecture, we stated that if $\varrho(x) = \alpha$ and $\varrho(y) = \beta$, then

$$\begin{aligned} \varrho(\{x, y\}) &= \max(\alpha, \beta) + 1, \\ \varrho(\mathrm{P}(x)) &= \alpha + 1, \\ \varrho(\{z \in x \,;\, \varphi(z)\} &\leq \alpha, \text{ and} \\ \varrho(\bigcup x) &\leq \alpha. \end{aligned}$$

You are allowed to use these results (cf. also Homework (18) on sheet #5).

(1) Let α be any ordinal. Being the powerset of another set is defined by the formula

$$\psi(p, x) \iff \forall z(z \in p \leftrightarrow z \subseteq x).$$

Suppose that x and $\mathrm{P}(x)$ are both elements of $\mathbf{V}_\alpha$. Argue that $\mathrm{P}(x)$ is the only element of $\mathbf{V}_\alpha$ such that

$$(\mathbf{V}_\alpha, \in) \models \psi(\mathrm{P}(x), x).$$

Why is this relevant for our argument that for limit ordinals α, $(\mathbf{V}_\alpha, \in)$ is a model of the powerset axiom?

(2) Check in detail that for limit ordinals α, $(\mathbf{V}_\alpha, \in)$ is a model of FST. We proved parts of this in the lecture and gave hints for other parts.

(3) In the lecture, we claimed that if $\alpha \geq \omega + 1$, then $(\mathbf{V}_\alpha, \in)$ is a model of the axiom of infinity. Think about what needs to be proved for this claim and prove it.

[*Note.* It is not enough to show that there is an element with infinitely many predecessors since that's not what the axiom of infinity says.]

(4) In the lecture, we sketched the argument that $\mathbf{V}_{\omega+\omega}$ does not satisfy the axiom scheme of Replacement. Give a careful argument for this claim by specifying precisely a functional formula Φ for which the scheme is violated.

(5) Generalise the argument from (4) to show the following claim:

> If α is an ordinal such that there is a $\beta < \alpha$ and a cofinal function $f : \beta \to \alpha$ that is definable over $\mathbf{V}_\alpha$ (i.e., there is a formula Φ such that $f(\gamma) = \delta$ if and only if $\mathbf{V}_\alpha \models \Phi(\gamma, \delta)$), then $\mathbf{V}_\alpha$ cannot be a model of the axiom scheme of Replacement.

(6) Explain why the assumption of *definability* was needed in your argument in (5).

(7) Let $X \subseteq \mathbf{V}_\omega$. We say that X is *closed under pairing* if for any $x, y \in X$, also $\{x, y\} \in X$. We say that X is *closed under union* if for any $x \in X$, also $\bigcup x \in X$. Characterise the subsets of $\mathbf{V}_\omega$ that are closed under both pairing and union.

(8) Remember the graph model constructions that we did in the first *Group Interaction*. We had *augmentation operations* that took an extensional graph $\mathbf{G} = (V, E)$ and extended it to a bigger extensional graph $\mathbf{G}' = (V', E')$ such that no new incoming edges for old vertices were constructed (i.e., if $v \in V$ and $(w, v) \in E'$, then $(w, v) \in E$; in other words, $\mathbf{G}'$ is an *end extension* of $\mathbf{G}$). Argue that all of the graphs obtained in the first *Group Interaction* using the *pairing closure* and the *power set closure* are isomorphic to a subset of $\mathbf{V}_\omega$.

[*Remark.* Now is a good moment to go back to the task sheet of *Group Interaction #1* and re-familiarise yourself with the notions of *Pairing Closure* and *Power Set Closure*, as well as the notions of an *extensional graph* and a *locally finite graph.*]

(9) Using (9) and (10), one can prove mathematically that if you start from the single irreflexive point, the methods from *Group Interaction #1* cannot produce a model that satisfies pairing, union, and the negation of powerset.

Before you start to prove this, first formulate a precise theorem: how do you specify "the methods from *Group Interaction #1*"? Which properties do you need the augmentation operation to have to make the argument work?

After you have made a precise mathematical claim, prove it.