Group Interaction #3

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one *group interaction* of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

In the third group interaction, we shall look at the *Cantor Normal Form* of ordinals. **You are allowed to use the rules of ordinal arithmetic and monotonicity that you will do on homework sheet #4.**

(1) *Ordinal subtraction.* Read and understand the following argument in detail:

> If $\beta \leq \alpha$, then there is a unique ordinal γ such that $\alpha = \beta + \gamma$.
>
> [Let η be least such that $\beta + \eta > \alpha$. (Why does this have to exist?) Observe that $\eta \neq 0$. Check that η cannot be a limit ordinal: if it is, then for all $\xi < \eta$, we have $\beta + \xi \leq \alpha$, but then $\beta + \eta = \bigcup\{\beta + \xi\,;\,\xi < \eta\} \leq \alpha$. Thus $\eta = \gamma + 1$. Check that $\alpha = \beta + \gamma$: by minimality of η, we have that $\beta + \gamma \leq \alpha < \beta + (\gamma + 1) = \beta + \eta$, thus $\beta + \gamma = \alpha$. Uniqueness follows from Homework (16d).]

(2) Note that the order of addition matters here: if $\alpha = \omega + 1$ and $\beta = \omega$, then there is no γ such that $\alpha = \gamma + \beta$. (Why?)

(3) Let $0 < \beta < \alpha$. Argue that there is a least η such that $\beta \cdot \eta > \alpha$ and that this η cannot be a limit ordinal.

[*Hint.* Follow the idea of the proof given in (1).]

(4) Prove the following statement of *Ordinal Division (with remainder):*

> Let $0 < \beta < \alpha$. Show that there are unique γ and ϱ such that $\alpha = \beta \cdot \gamma + \varrho$ and $\varrho < \beta$.

[*Hint.* Use (3).]

(5) Note, as in (2), that the order of multiplication matters here: find examples of ordinals $\beta < \alpha$ such that it is not possible to write α as $\gamma \cdot \beta + \varrho$ with $\varrho < \beta$.

(6) *Ordinal Logarithm (with remainders).* Let $1 < \beta < \alpha$. Show that there are unique γ, ϱ_0, and ϱ_1 such that

(a) $\alpha = \beta^\gamma \cdot \varrho_0 + \varrho_1$,
(b) $\varrho_0 < \beta$, and
(c) $\varrho_1 < \beta^\gamma$.

(7) Let δ be an ordinal. A finite sequence $(\delta_0, ..., \delta_n)$ with $\delta_0 \geq \delta_1 \geq ... \geq \delta_n$ is called a *Cantor Normal Form of* δ if

$$\delta = \omega^{\delta_0} + \cdots + \omega^{\delta_n}.$$

Prove that every ordinal $\delta > 0$ has a Cantor Normal Form.

[*Hint.* Apply (6) with $\beta = \omega$ and $\alpha = \delta$ to obtain γ, ϱ_0, and ϱ_1 and iterate. Why does this process terminate after a finite number of steps? How does n relate to that finite number of steps?]

(8) Please skip this during the Group Interaction, but keep it in mind when you return to this sheet afterwards (e.g., in the preparation for the exam): it can be shown that the Cantor Normal Form of an ordinal is unique.

(9) Determine the Cantor Normal Form of the following ordinals:

(a) $1 + \omega$,

(b) $2 \cdot \omega$,

(c) $\omega \cdot 2$, and

(d) $(\omega + 2) \cdot (\omega \cdot 2 + 2)$, and

(e) $(\omega + 2)^{\omega+2}$.

(10) An ordinal γ is called *selfnormal* if it is its own Cantor Normal Form, i.e., $\omega^\gamma = \gamma$. Can you find a selfnormal ordinal? Can you find a countable selfnormal ordinal?

(11) The Cantor Normal Form in (7) uses the base ω. Let β be any ordinal. Formulate and prove a version of the Cantor Normal Form theorem for the base β. For which ordinals β can you prove the theorem?

(12) An ordinal γ is called a *gamma number* (or *principal number of addition*) if it is closed under addition, i.e., for all $\alpha, \beta \in \gamma$, we have $\alpha + \beta \in \gamma$. Show that $\gamma \neq 0$ is a gamma number if and only if there is a ξ such that $\gamma = \omega^\xi$.

(13) An ordinal δ is called a *delta number* (or *principal number of multiplication*) if it is closed under multiplication, i.e., for all $\alpha, \beta \in \gamma$, we have $\alpha \cdot \beta \in \gamma$. Show that $\delta \notin \{0, 1\}$ is a delta number if and only if there is a ξ such that $\delta = \omega^{(\omega^\xi)}$.