Group Interaction #2

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one *group interaction* of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

In the second group interaction, we shall look at the details of the re-construction of the integers, rationals, and real numbers in Zermelo set theory Z.

(1) A structure $\mathbf{A} = (A, +, 0)$ is called *cancellative abelian monoid* if for all $a, b, c \in A$, we have:

$$
\begin{aligned}
a + (b + c) &= (a + b) + c, \\
a + b &= b + a, \\
a + 0 &= a, \\
0 + a &= a \text{ and} \\
\text{if } a + b = a + c, &\text{ then } b = c.
\end{aligned}
$$

The last condition is called the *cancellation law.* If $\mathbf{A}$ is a cancellative abelian monoid, we define a relation $\approx$ on $A \times A$ by $(a, b) \approx (a', b')$ if and only if $a + b' = b + a'$.

Show that $\approx$ is an equivalence relation. Highlight the use of the cancellation law in your argument and consider what happens if the monoid $\mathbf{A}$ is not cancellative.

(2) Presumably, you did the argument in (1) without thinking about set theory. Briefly confirm that the statement of (1) can be reformulated as a theorem of FST.

(3) Writing $[a, a']_\approx$ for the $\approx$-equivalence class of the pair (a, a'), define an operation $+$ on the set of all $\approx$-equivalence classes $G := (A \times A)/\approx$ such that $((A \times A)/\approx, +, [0, 0]_\approx)$ is a group and the map

$$i : A \to G : a \mapsto [a, 0]_\approx$$

is a structure-preserving injection.

(4) As in (2), check that the statement of (3) can be expressed as a theorem of FST.

(5) Now move to Zermelo set theory Z. Briefly convince yourselves that $(\mathbb{N}, +, 0)$ is a cancellative abelian monoid. Observe that we proved some of the statements (including homework question (11) on homework sheet #3) and check which other statements would have to be proved. Do not do the proof right now since it would take too much time in the group interaction, but keep it in the back of your mind that it would be good to check these later.

(6) Working in Z, use the construction from (3) and apply it to $(\mathbb{N}, +, 0)$ to give a precise definition of "*the integers*" in Z.

(7) What are the integers that you just defined as set theoretic objects? E.g., what is the neutral element of the integers as a set? Do we get $\mathbb{N} \subseteq \mathbb{Z}$ with this definition?

[*Hint.* The neutral element (actually, any element) is a set of pairs of natural numbers.]

(8) In which sense is the definition of "the integers" given in (6) unique?

[*Hint.* Think of the sketch of an alternative construction of the integers given in the third lecture.]

(9) Discuss how this construction applied to $\mathbb{Z}$ can be used to define "*the rationals*" in Zermelo set theory Z. Again, discuss in which sense this defines a unique set theoretic object.

[*Hint.* Use multiplication instead of addition. One of the integers breaks the cancellation law for multiplication, so you need to exclude it.]

(10) The next definition once more makes sense in FST: If $\mathbf{T} = (T, <)$ is a strict total order, we call a pair (L, R) with $L, R \subseteq T$ a *Dedekind cut* if

  (a) L is a proper initial segment of T, i.e., $L \neq T$ and if $\ell \in L$ and $t < \ell$, then $t \in L$;
  (b) R is a proper final segment of T, i.e., $R \neq T$ and if $r \in R$ and $r < t$, then $t \in R$;
  (c) R and L partition T, i.e., $R \cap L = \varnothing$ and $R \cup L = T$;
  (d) L does not have a largest element.

Briefly confirm that this can be expressed in the language of set theory.

(11) We write $\mathrm{Ded}(\mathbf{T})$ for the set of Dedekind cuts of $\mathbf{T}$ and define an order on $\mathrm{Ded}(\mathbf{T})$ by $(L, R) < (L', R')$ if and only if $L \subsetneq L'$. Show that $(\mathrm{Ded}(\mathbf{T}), <)$ is a linear order.

[*Remark.* If you are still unsure about the implicit uses of Separation that we are now making all the time, write down the definition of $\mathrm{Ded}(\mathbf{T})$ in (an approximation to) the language of set theory to convince yourself that these definitions can really all be made in models of FST.]

(12) If $\mathbf{T} = (T, <)$ is a strict total order and $Z \subseteq T$, we say that $x \in T$ is an *upper bound* of Z if for all $z \in Z$, we have $z \leq x$. We say that Z is *bounded from above* if it has an upper bound. We say that $x \in T$ is the *supremum* of Z if it is an upper bound and for all upper bounds b, we have $x \leq b$.

The strict total order $\mathbf{T}$ is called *complete* if every subset bounded from above has a supremum. Check that $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are complete, but $(\mathbb{Q}, <)$ is not.

(13) Show that for every linear order $\mathbf{T}$, the order $(\mathrm{Ded}(\mathbf{T}), <)$ is complete.

(14) Show that the map $q \mapsto (\{x \in \mathbb{Q}\,;\, x < q\}, \{x \in \mathbb{Q}\,;\, x \geq q\})$ is an order-preserving injection from $\mathbb{Q}$ into $\mathrm{Ded}(\mathbb{Q})$ whose image lies dense in $\mathrm{Ded}(\mathbb{Q})$, i.e., for any $x < y$ there is a z between them that is in the image.

(15) Use (14) to define "*the real numbers*" in Zermelo set theory Z. Again, discuss in which sense this defines a unique set theoretic object.

(16) How would you define the operations of addition and multiplication on your real numbers (in a way that extends the operations on $\mathbb{Q}$?