Group Interaction #11

MasterMath: Set Theory
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Every week, there will be one group interaction of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

This group interaction deals with further properties of the partial order used in class. We denote the partial order as $\mathrm{Fn}(\omega_2 \times \omega, 2)$.
Given a model M and an M-generic filter G on $\mathrm{Fn}(\omega_2^M \times \omega, 2)$ the resulting map from ω_2^M to $\mathcal{P}^N(\omega)$ depends on G of course. Therefore we write it as f_G, so $f_G(\alpha) = \{n : \bigcup G(\alpha, n) = 1\}$.

(1) Remind yourself of the definition of a dense subset in $\mathrm{Fn}(\omega_2 \times \omega, 2)$: a set D is dense if for every p there is $q \leqslant p$ that is in D.
Check that the whole set $\mathrm{Fn}(\omega_2 \times \omega, 2)$ is dense, and the empty set is not.

(2) Two functions p and q in $\mathrm{Fn}(\omega_2 \times \omega, 2)$ are *incompatible* if there is no function r such that $r \leqslant p, q$. Given a nonempty p find a q that is incompatible with p.

(3) Show: if $p \in \mathrm{Fn}(\omega_2 \times \omega, 2)$ then $\{p\}$ is not dense.

(4) In class we used $E_{\alpha,\beta} = \{p : (\exists n)(p(\alpha, n) \neq p(\beta, n))\}$ to show that $f_G(\alpha) \neq f_G(\beta)$ whenever $\alpha \neq \beta$. Study the proof that $E_{\alpha,\beta}$ is dense:

> Let p be given and take $n \in \omega$ such that $\langle \alpha, n\rangle$ and $\langle \beta, n\rangle$ are not in $\operatorname{dom} p$. Define q by $\operatorname{dom} q = \operatorname{dom} p \cup \{\langle \alpha, n\rangle, \langle \beta, n\rangle\}$ and: $q(\gamma, m) = p(\gamma, m)$ if $\langle \gamma, m\rangle \in \operatorname{dom} p$, and $q(\alpha, n) = 1$ and $q(\beta, n) = 0$.

Why is the choice possible? Why is q in $E_{\alpha,\beta}$? Why do we have $q \leqslant p$?

(5) Let $x \in M$ be an infinite subset of ω. Prove that for every n the set

$$D_n = \{p : (\exists k, l \in x)(k, l \geqslant n \wedge p(0, k) \neq p(0, l))\}$$

is dense.

(6) Deduce that if $x \in M$ is an infinite subset of ω then $x \cap f_G(0)$ and $x \setminus f_G(0)$ are both infinite.

For the next few exercises we fix distinct α and β in ω_2.

(7) What dense set would you use to show that there is an $n > 100$ such that $n \in f_G(\alpha) \setminus f_G(\beta)$?

(8) What dense set would you use to show that there is an $n > 100$ such that $n \in f_G(\alpha) \cap f_G(\beta)$?

(9) What dense set would you use to show that there is an $n > 100$ such that $n \notin f_G(\alpha) \cup f_G(\beta)$?

(10) Prove that all four sets $f_G(\alpha) \setminus f_G(\beta)$, $f_G(\beta) \setminus f_G(\alpha)$, $f_G(\alpha) \cap f_G(\beta)$ and $\omega \setminus (f_G(\alpha) \cup f_G(\beta))$ are infinite.

(11) Let A and B be disjoint finite subsets of ω_2. Show that

$$\bigcap_{\alpha \in A} f_G(\alpha) \setminus \bigcup_{\beta \in B} f_G(\beta)$$

is infinite.

(12) Explain how (11) can be used that the sets $f_G(\alpha)$ are (completely) independent of each other: given α there is no finite combination of unions, intersections and complements that uses the sets $\{f_G(\beta) : \beta \neq \alpha\}$ and is equal to $f_G(\alpha)$.