Group Interaction #1

MasterMath: Set Theory
2021/22: 1st Semester
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Every week, there will be one *group interaction* of roughly one hour. The group interactions take place remotely via Zoom. A group interaction consists of two students who work together on a work sheet in the presence of one of the two teaching assistants (Steef Hegeman or Robert Paßmann). A group does not have to cover the entire work sheet. If you do not finish the work sheet, feel free to return to it later or in the preparation of the exam.

Students are expected to actively participate in these group interaction sessions each week. The *group interaction score* is the number of times a student actively participated in one of the group interaction sessions (the maximum score is **10**). [The score of students who are sitting the *Rudiments* exam is multiplied by two (with a maximum score of **10**).]

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In the first group interaction, we shall construct infinite graph models of weak set theories. These constructions happen in naïve set theory and you are allowed to use all of the tools of ordinary mathematics (i.e., recursion and induction) to prove things about your models.

(1) Let $\mathbf{G} = (V, E)$ be any directed graph. If $v \in V$, we write $\mathrm{pred}_{\mathbf{G}}(v) := \{w \in V\,;\, w\ E\ v\}$ for the set of $\mathbf{G}$-predecessors of v. We write $[V]^{\leq 2}$ for the set of all subsets of V having at most two vertices. If $Z \in [V]^{\leq 2}$, we say that Z is *covered in* $\mathbf{G}$ if there is some $v \in V$ such that $\mathrm{pred}(v) = Z$. Otherwise, we say that Z is *uncovered in* $\mathbf{G}$.

Draw an example (directed) graph with four vertices. It has $\binom{4}{1} = 4$ one-vertex subsets and $\binom{4}{2} = 6$ two-vertex subsets. Check which of these are *covered* and *uncovered.*

(2) The directed graph $\mathrm{p}(\mathbf{G}) := (V^*, E^*)$ is called the *pairing augmentation of* $\mathbf{G}$ if V^* consists of all of the vertices of V plus a set of new vertices V^+ such that each new vertex $v \in V^+$ corresponds to exactly one set $Z \in [V]^{\leq 2}$ that is uncovered in $\mathbf{G}$ with $\mathrm{pred}_{\mathrm{p}(\mathbf{G})}(v) = Z$. Furthermore, for each $v \in V$, $\mathrm{pred}_{\mathbf{G}}(v) = \mathrm{pred}_{\mathrm{p}(\mathbf{G})}(v)$.

Take your example graph from (1) and draw its pairing augmentation.

(3) Repeat the process: now take the pairing augmentation of your original four-vertex graph from (2), and draw its pairing augmentation.

(4) Given any directed graph $\mathbf{G}$, we define by recursion

$$\mathbf{G}_0 := \mathbf{G} \text{ and}$$
$$\mathbf{G}_{n+1} := \mathrm{p}(\mathbf{G}_n).$$

Write $\mathbf{G}_n := (V_n, E_n)$ and define $V_\infty := \bigcup_{n\in\mathbb{N}} V_n$ and $E_\infty := \bigcup_{n\in\mathbb{N}} E_n$. We call the directed graph $\mathbf{G}_\infty := (V_\infty, E_\infty)$ the *pairing closure of* $\mathbf{G}$.

Show that for every directed graph $\mathbf{G}$, $\mathbf{G}_\infty$ is a model of the pairing axiom.

(5) A directed graph $\mathbf{G}$ is called *extensional* if it is a model of the axiom of extensionality. Show that if $\mathbf{G}$ is extensional, then so is $\mathrm{p}(\mathbf{G})$.

(6) Show that if $\mathbf{G}$ is extensional, then so is $\mathbf{G}_\infty$.

(7) Let now $\mathbf{H} := (\{e\}, \varnothing)$, the directed graph with a single vertex e and apply the pairing closure operation to it. First get a feeling for the construction by drawing $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, and $\mathbf{H}_4$.

(8) By (4) and (5), $\mathbf{H}_\infty$ is a model of pairing and extensionality. Check whether the union axiom and the power set axiom hold in $\mathbf{H}_\infty$.

(9) Formulate the result from (8) as a statement about strength of axiom systems, i.e., in the form "T is strictly stronger than S".

(10) Show that $\mathbf{H}_\infty$ satisfies the axiom scheme of separation.

[*Hint.* Show first that for every vertex $v \in \mathbf{H}_\infty$, we have that $\text{pred}(v)$ has at most two elements. This helps us to reduce the claim of the axiom scheme of separation to something more manageable.]

(11) Let $\mathbf{H}'$ be the directed graph with two vertices and no edges. (As discussed in Lecture I, this does not satisfy Extensionality.) Check the other axioms in $\mathbf{H}'_\infty$ and formulate your findings in terms of strength of axiom systems as in (9).

(12) Is it possible to use the construction of pairing closure to get a directed graph that does not satisfy the axiom scheme of separation?

(13) We are now going to define a different graph construction: again, let $\mathbf{G} = (V, E)$ be an arbitrary directed graph.

For each $v \in V$, we can consider $P(v) := \{w \in V\,;\, w \text{ is a } \mathbf{G}\text{-subset of } v\} \subseteq V$. We call a vertex v *handled in* $\mathbf{G}$ if there is some $w \in V$ such that $\text{pred}_{\mathbf{G}}(w) = P(v)$. Otherwise, we say that v is *unhandled in* $\mathbf{G}$.

Check the status of handled and unhandled vertices in your example graph from (1).

(14) The *power set augmentation of* $\mathbf{G}$ is defined in the same way as the pairing augmentation, but the new vertices in V^+ have as predecessors precisely the sets $P(v)$ for a vertex v that was unhandled in $\mathbf{G}$. We write $\text{pow}(\mathbf{G})$ for the power set augmentation of $\mathbf{G}$ and define the *power set closure* in the same way.

Show that if $\mathbf{G}$ is extensional, then the power set closure of $\mathbf{G}$ is extensional.

(15) Show that if $\mathbf{G}$ is extensional and locally finite (i.e., every vertex has only finitely many predecessors), then the power set closure of $\mathbf{G}$ satisfies the power set axiom.

[*Hint.* Notice that extensionality means that if v has n predecessors, there can be at most 2^n many elements of $P(v)$. Observe that this means that after finitely many stages of the construction, all of them have appeared.]

(16) Using $\mathbf{H} := (\{e\}, \varnothing)$ from (7), consider its power set closure. First, draw stages 1 to 4 in that construction to get a feeling for the construction.

(17) Check the validity of the union axiom, and the axiom scheme of separation in the power set closure of $\mathbf{H}$ and express your result in terms of strength of axiom systems as in (9).

(18) Think about whether the techniques of pairing closure and power set closure could be combined to give models that satisfy both the pairing axiom and the power set axiom.