Understanding & Explanation Task U6

MasterMath: Set Theory
2021/22: 1st Semester
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Deadline for Understanding & Explanation Task U6: Monday, 6 December 2021, 2pm. Please hand in via the `elo` webpage as a single pdf file.

Understanding & Explanation tasks (U).

Marking Scheme.

An answer will be considered **good** if all three criteria are satisfied. These answers will get full points (i.e., $\mathbf{3\frac{1}{2}}$ **points**).

It will be considered **satisfactory** if it has minor deficiencies in some of the three criteria. Satisfactory answers will get **3 points**.

It will be considered unsatisfactory if it has a major deficiency in either correctness or comprehensivity. Unsatisfactory answers will get either **2 points**, **1 point**, or **0 points**, depending on the flaws.

Task U6: We went to a lot of trouble to create a *countable* transitive set M that would satisfy our finitely many axioms.
We saw one advantage of this already: it is easy to construct M-generic filters.
The drawback (to some) is that the whole universe of M is countable and does not contain all subsets of ω, not the real ω_1 etc.

Explain to those doubters that the perceived drawback is imaginary and that countable transitive structures are adequate for our purposes.

Solution

The first thing to explain is that when discussing (un)provability is that we consider Set Theory to be a first-order theory in the language $\{\in, =\}$ (plus the usual logical symbols of course). The (un)provability is about formal proofs that use the axioms of ZFC as their starting points.
Therefore our statements are actually of the form "There is no formal proof in first-order logic that uses the axioma of ZFC as its starting assumptions".
Therefore we can you all results from first-order logic in our effort to establish unprovability. The completeness results tell us that to show that a statement is unprovable it is sufficient to construct an interpretation of our language that satisfies the axioms of ZFC but not the statement we want to show unprovable.
The Lẅenheim-Skolem theorem implies that, because our language is countable, any interpretation has a countable substructure that satisfies the same sentences as the initial one.
In our case having a countable structure that satisfies the axioms of ZFC simply tells us that countability, and in general cardinality, is not an absolute notion. If $(|A| = \aleph_1)^M$ then this tells us that for every $f \in M$ that is a map from ω to A we know that it is not surjective.