ANALYTIC DETERMINACY

Notation. Fix a bijection $i \mapsto s_i$ from $\omega \rightarrow \omega^\omega$ such that if $s_i \subseteq s_j$, then $i \leq j$. (This implies that $lh(s_i) \leq i$.) Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree, $x \in \omega^\omega$, and $s \in \omega^{<\omega}$. Then we let

$$
T_s := \{ t \in \omega^{<\omega} ; (s \upharpoonright lh(t), t) \in T \},
$$

$$
T_x := \{ t \in \omega^{<\omega} ; (x \upharpoonright lh(t), t) \in T \} = \bigcup_{n \in \mathbb{N}} T_x \upharpoonright n,
$$

$$
K_s := \{ i \leq lh(s) ; s_i \in T_s \},
$$

$$
K_x := \{ i \in \omega ; s_i \in T_x \} = \bigcup_{n \in \mathbb{N}} K_x \upharpoonright n.
$$

We note that $T_s$ is a tree of finite height (every element $t \in T_s$ has length $\leq lh(s)$) and that $K_s$ is a finite set. We observe that $T_x = \{ s_i ; i \in K_x \}$ (but, in general, $T_x \nsubseteq \{ s_i ; i \in K_x \}$).

We remember that if $A \in \Pi^1_1$, then there is a tree $T$ on $\omega \times \omega$ such that

$$
x \in A \text{ if and only if } (T_x, <_{KB}) \text{ is wellfounded}
$$

if and only if $(T_x, <_{KB})$ is wellordered

if and only if there is an order preserving map from $(T_x, <_{KB})$ to $(\omega_1, <)$

where $<_{KB}$ is the Kleene-Brouwer order on $\omega^{<\omega}$. For any $s \in \omega^{<\omega}$, we write $<_s$ for the order induced by the Kleene-Brouwer order on $\omega^{<\omega}$ on $K_s$, i.e., $i <_s j$ if and only if $s_i <_{KB} s_j$. Note that since $K_s$ is finite, $(K_s, <_s)$ is a (finite) wellorder.

Let $S$ be any tree on $\omega$ and $\kappa$ be an uncountable cardinal. A function $g : \omega \rightarrow \kappa$ is called a KB-code for $S$ if for all $i$ and $j$ such that $s_i, s_j \in S$, we have that $s_i <_{KB} s_j \iff g(i) < g(j)$. Clearly, there is an order preserving map from $(S, <_{KB})$ to $(\omega_1, <)$ if and only if there is a KB-code for $S$, so we can add the following equivalence to the above characterisation of $\Pi^1_1$ sets:

$$
x \in A \text{ if and only if there is a KB-code for } T_x
$$

Shoenfield’s Theorem. We first prove a tree representation theorem for $\Pi^1_1$ sets.

Theorem 1 (Shoenfield). If $\kappa$ is uncountable, then every $\Pi^1_1$ set is $\kappa$-Suslin.

Proof. Let $A \in \Pi^1_1$ and let $T$ be a tree on $\omega \times \omega$ such that $x \in A$ if and only if there is a KB-code for $T_x$. Let $M$ be the set of all partial functions from $\omega$ into $\kappa$ with finite domain. Note that $|M| = \kappa$, so it is sufficient to show that $A$ is $M$-Suslin. If $s \in \omega^{<\omega}$ and $u \in M^{<\omega}$ such that $lh(h) \leq lh(s)$, we say that $u$ is coherent with $s$ if

1. for all $i < lh(u)$, we have that dom($u_i$) = $K_{s|i}$,
2. for all $i < lh(u)$, $u(i)$ is an order preserving map from $(K_{s|i}, <_{KB})$ into $(\kappa, <)$, and
3. for $i \leq j$, we have that $u_i \subseteq u_j$.

We now define the Shoenfield tree on $\omega \times M$ by $\hat{T} := \{ (s, u) ; u \text{ is coherent with } s \}$ and claim that $A = \text{p}[\hat{T}]$.

$\subseteq$: If $x \in A$, then let $g : \omega \rightarrow \kappa$ be a KB-code for $T_x$ and define $u(i) := g|K_{x|i}$. By definition, $u|n$ is coherent with $x|n$ for all $n$, and so $(x, u) \in [\hat{T}]$.

$\supseteq$: If $x \in \text{p}[\hat{T}]$, find $u \in M^{<\omega}$ such that $(x, u) \in [\hat{T}]$; this means that for each $n$, $u|n$ is coherent with $x|n$.

As noted above, we have that $T_x = \{ s_i ; i \in K_x \} = \{ s_i ; 3 \upharpoonright i \in \text{dom}(u(n)) \}$. We define $\hat{u} := \bigcup \{ u(i) ; i \in \omega \}$.

By coherence, $\hat{u}$ is a function from $K_x$ to $\kappa$; now we define

$$
g : \omega \rightarrow \kappa : n \mapsto \begin{cases} 
\hat{u}(n) & \text{if } n \in K_x \\
0 & \text{otherwise}
\end{cases}.
$$
We claim that $g$ is a KB-code for $T_x$ whence $x \in A$: Suppose not, then there are $i$ and $j$ such that $s_i, s_j \in T_x$ and $s_i \not<_{KB} s_j \not\in g(i) < g(j)$. Since $i, j \in K_x$, find $n$ large enough such that $i, j \in K_x|n$. By definition $g|K_x|n = u(n)$. But this means that $u(n)$ is not an order preserving map from $(K_x|n, <_{KB})$ into $(\kappa, \prec)$, violating condition (3) of coherence. Q.E.D

**Measurable Cardinals.** Let $X$ be a set. A non-empty family $U \subseteq \wp(X)$ is called a **ultrafilter over** $X$ if for any $A, B \subseteq X$, we have that

1. if $A, B \in U$, then $A \cap B \in U$,
2. if $A \in U$ and $B \supseteq A$, then $B \in U$, and
3. if $A \notin U$, then $X\setminus A \in U$.

We say that an ultrafilter is **non-trivial** if it does not contain any finite sets and if $\kappa$ is any cardinal, it is called $\kappa$-**complete** if it is closed under intersections of size $< \kappa$. Note that $\omega$-completeness follows from (1).

A non-trivial $\kappa$-complete ultrafilter cannot contain any sets of size $< \kappa$.

[If $|A| = \lambda < \kappa$, then for each $a \in A, \{a\} \notin U$, so by (3), $X\setminus \{a\} \in U$, but then by $\kappa$-completeness, $X\setminus A = \bigcap\{X\setminus \{a\} : a \in A\} \in U$. If now $A \subseteq U$, then $\varnothing = A \cap X\setminus A \in U$. Contradiction to non-triviality.]

An uncountable cardinal $\kappa$ is called **measurable** if there is a $\kappa$-complete non-trivial ultrafilter on $\kappa$. The Axiom of Choice implies that there are non-trivial ultrafilters on $\omega$; as mentioned, they are $\omega$-complete, so $\aleph_0$ technically satisfies the conditions of the definition. The existence of uncountable measurable cardinals cannot be proved in ZFC and is a so-called **large cardinal axiom**. More precisely, if MC stands for “there is a measurable cardinal”, then for every model $M \models ZFC + MC$, I can find a submodel $N \subseteq M$ such that $N \models ZFC + \neg MC$.

Being measurable has interesting consequences for the combinatorics on $\kappa$. We are going to use one of them in our proof of analytic determinacy. As usual, we denote by $[\kappa]^n$ the set of $n$-element subsets of $\kappa$. A function $f : [\kappa]^n \to \omega$ is called an $n$-**colouring** and a set $H$ is called **homogeneous for** $f$ if $f|H^n$ is constant. We call $f$ a **finite colouring** if it is an $n$-colouring for some natural number $n \in \mathbb{N}$.

**Theorem 2** (Rowbottom). If $\kappa$ is measurable, then for every countable set $\{f_s : s \in S\}$ of finite colourings, there is a set $H$ of size $\kappa$ that is homogeneous for all colourings $f_s$.

In our proof of analytic determinacy, we are only going to use Rowbottom’s Theorem, no other properties of measurable cardinals: so, for our purposes, one could take the statement of Rowbottom’s Theorem as the assumption for analytic determinacy in the next section.

**Analytic Determinacy.** If $\Gamma$ is a boldface pointclass, then Det($\Gamma$) is equivalent to Det($\bar{\Gamma}$). Thus, analytic determinacy and co-analytic determinacy are equivalent.

**Theorem 3** (Martin, 1969/70). If there is a measurable cardinal, then every co-analytic set is determined.

**Proof.** Let $\kappa$ be a measurable cardinal and $A \in \Pi_1^1$. We aim to show that the game $G(A)$ is determined. By (the proof of) Shoenfield’s Theorem, we know that there is a tree $\bar{T}$ on $\omega \times M$ such that $A = p[\bar{T}]$. (Remember that $\Gamma$ was the set of partial functions from $\omega$ to $\kappa$ with finite domain.) We are going to define a (determined) game $G_{aux}(\bar{T})$ based on the Shoenfield tree and show that a winning strategy for either player in $G_{aux}(\bar{T})$ can be transformed into a winning strategy for the same player in the original game $G(A)$. This proves the theorem.

In the **auxiliary game**, player I plays elements of $\omega \times M$ and player II plays elements of $\omega$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>$x_0, u_0$</th>
<th>$x_1$</th>
<th>$x_2, u_1$</th>
<th>$x_3$</th>
<th>$x_4, u_2$</th>
<th>$x_5$</th>
<th>$x_6, u_3$</th>
<th>$\cdots$</th>
<th>$x_7$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$x_0$</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
<td>$x_5$</td>
<td>$x_6$</td>
<td>$u_3$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>II</td>
<td>$u_0$</td>
<td>$u_1$</td>
<td>$u_2$</td>
<td>$u_3$</td>
<td>$u_4$</td>
<td>$u_5$</td>
<td>$u_6$</td>
<td>$u_7$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

We obtain a sequence $x \in \omega^\omega$ with $x(n) := x_n$ and a sequence $u \in M^\omega$ with $u(n) := u_n$. Player I wins $G_{aux}(\bar{T})$ if $(x, u) \in [\bar{T}]$. Note that $G_{aux}(\bar{T})$ is a closed game on $\omega \times M$, thus by the Gale-Stewart Theorem, it is determined.

Let us make a number of observations about the relationship between the original game $G(A)$ and the auxiliary game $G_{aux}(\bar{T})$. We call the moves $u_i$ **auxiliary moves**. If $p$ is a position in the auxiliary game (i.e., a finite sequence of elements of $\omega$ and elements of $M$ in the right order), then we can define a position $p^*
Lemma 4. If player I has a winning strategy in $G_{\text{aux}}(\hat{T})$, then they have a winning strategy in $G(A)$.

Proof. Suppose $\sigma$ is a winning strategy in $G_{\text{aux}}(\hat{T})$ and $\tau$ is any strategy for player II in the original game. As just mentioned, then $\tau_*$ is the version of that strategy in $G_{\text{aux}}(\hat{T})$. Since $\sigma$ is winning, we know that $\sigma * \tau_* = (x, u) \in [\hat{T}]$. Define a strategy $\sigma^*$ in the original game as follows: while player II plays natural number moves according to $\tau$, you produce the auxiliary play $\sigma * \tau_*$ on an auxiliary board. If that auxiliary game tells you to produce a position $p$ by your next move, then you produce the move $p'$ in the original game. Then $\sigma^* \tau = x$, and thus $x \in \rho[\hat{T}] = A$, so $\sigma^*$ is winning. Q.E.D

Lemma 5. If player II has a winning strategy in $G_{\text{aux}}(\hat{T})$, then they have a winning strategy in $G(A)$.

Proof. Let $s \in \omega^{<\omega}$. Let $k_s := |K_s|$. If $Q \in [\kappa]^{k_s}$, then there is a unique order preserving map $w : (K_s, <s) \to (Q, <)$. Let $u_i^\kappa Q = w[K_s]$. Then $(u_i^\kappa Q : i < \text{lh}(s))$ is coherent with $s$. Thus, if you fix some $Q \in [\kappa]^{k_s}$, you can transform a position $s$ in the original game into a position $s_* Q$ in the auxiliary game in such a way that the auxiliary moves produce $Q$ as the range and form a sequence coherent with the position $s$.

Let now $\tau$ be a strategy for player II in the auxiliary game. For each $s \in \omega^{<\omega}$, we define a $k_s$-colouring $f_s : [\kappa]^{k_s} \to \omega$ by $f_s(Q) = \tau(s_* Q)$: we colour the $k_s$-element subsets of $\kappa$ by the answer that the strategy $\tau$ gives to the position $s$ augmented via $Q$ in the sense given above. By Rowbottom’s theorem, there is a set $H \subseteq \kappa$ of size $\kappa$ that is homogeneous for all functions $f_s$, i.e., if $Q, Q' \in [H]^{k_s}$, then $\tau(s_* Q) = f_s(Q) = f_s(Q') = \tau(s_* Q)$, so the answer of the strategy $\tau$ does not depend on the set $Q$ as long as it is a subset of $H$. In particular, we can take the simplest imaginable subset of $H$ with $k_s$ elements: let $Q_{H,s}$ be the set consisting of the first $k_s$ many elements of $H$.

Now, we define a strategy $\tau_H$ for player II in the original game by $\tau_H(s) := \tau(s_* Q_{H,s})$. (Note that the precise choice of the set $Q_{H,s}$ is irrelevant in this definition by homogeneity, since $f_s(Q_{H,s}) = f_s(Q)$ for any $Q \in [H]^{k_s}$.)

We prove that if $\tau$ was winning in the auxiliary game, then $\tau_H$ is winning in the original game. Suppose not, so there is a counterstrategy $\sigma$ such that $x := \sigma * \tau_H \in A$. This means (since $H$ is uncountable) that there is an orderpreserving map from $(T_x, <_{\text{KB}})$ to $(H, <)$ giving rise to a KB-code $g : \omega \to H$ for $T_x$. Using the KB-code $g$, we can now define $u_i := g[K_{x}]$ and consider the play of the auxiliary game

\[
\begin{array}{c|ccccccc}
I & x_0, u_0 & x_2, u_1 & x_4, u_2 & x_6, u_3 & \cdots \\
II & x_1 & x_3 & x_5 & x_7 & \cdots \\
\end{array}
\]

producing $(x, u) \in [\hat{T}]$. We claim that this is a play according to $\tau$, so we need to show that for every $i \in \mathbb{N}$, the play by player II is the $\tau$-answer to the previous position, i.e., $x_{2i+1} = \tau(x_0, u_0, x_1, ..., x_{2i}, u_i)$. Fix $i \in \mathbb{N}$ and consider $Q := \text{ran}(u_i) \subseteq H$. Then we have that $(x_0, u_0, x_1, ..., x_{2i}, u_i) = (x[2i + 1]_* Q).$ We see that

\[
x_{2i+1} = \tau_H(x[2i + 1]) = \tau((x[2n + 1]_* Q_{H,s}))) = \tau((x[2n + 1]_* Q)) = \tau(x_0, u_0, x_1, ..., x_{2i}, u_i),
\]

so the above play is a play according to $\tau$. But that is a contradiction, since $\tau$ was winning for player II, and so $(x, u) \notin [\hat{T}]$. Q.E.D

Lemmas 4 & 5 together with the fact that $G_{\text{aux}}(\hat{T})$ was determined (since it is a closed game) imply that $G(A)$ is determined. Q.E.D