Capita Selecta: Set Theory

NINTH LECTURE (Part I)

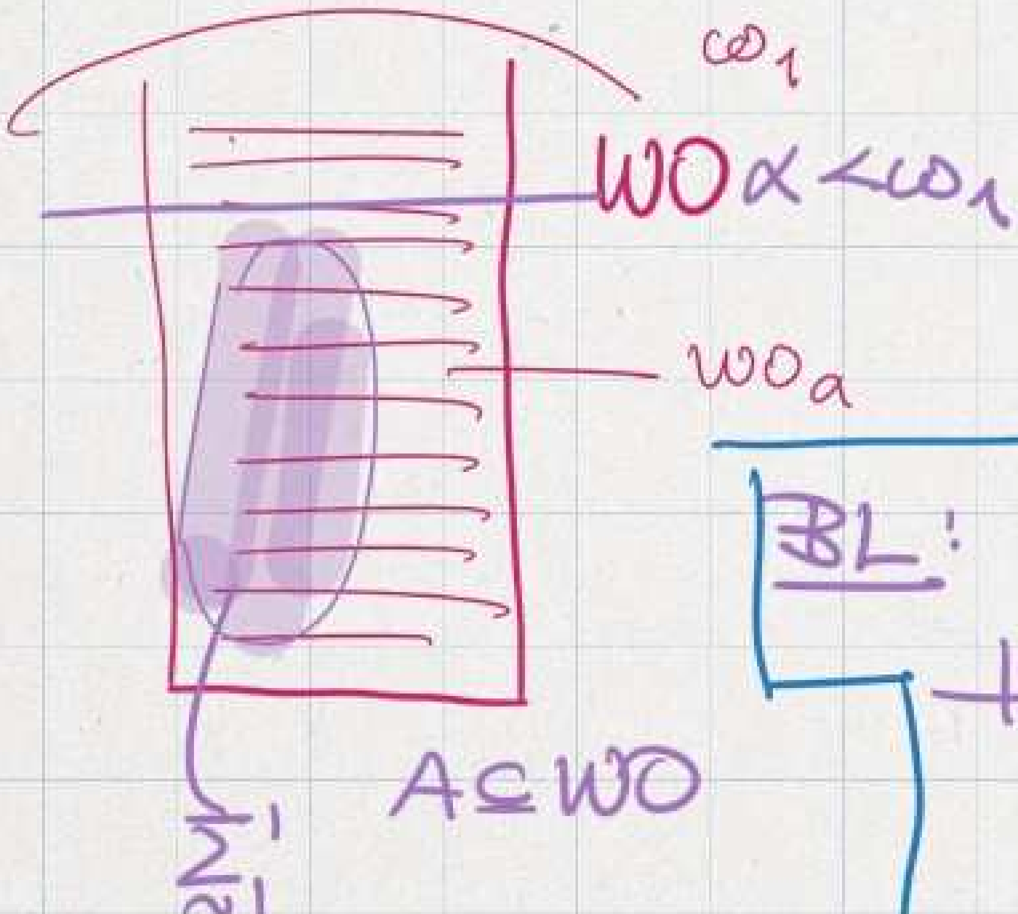
Proof of the Boundedness Lemma.

WO_α $\quad \underline{WO} := \{x \,;\, (\mathbb{N}, E_x) \text{ is wellorder}\}$

WO^*_α $\quad \underline{WO}^* := \{x \,;\, (\mathrm{fld}(x), E_x) \text{ is wellorder}\}$

WF_α $\quad WF := \{x \,;\, (\mathrm{fld}(x), E_x) \text{ is wellfdd}\}$

$$WO^* = WF \cap LO^*$$



$$WO_\alpha = \{x \in WO \,;\, \underline{\|}x\underline{\|} = \alpha\}$$

either the unique ord. iso / or the rank

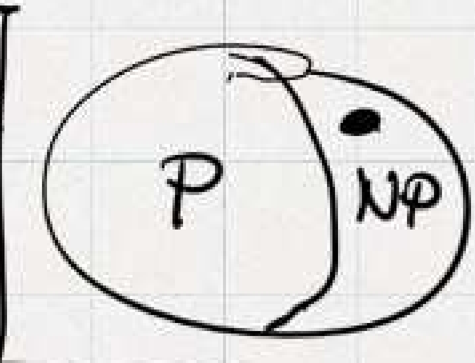
$\underline{BL}$: If $A \subseteq WO$ is $\boldsymbol{\Sigma}^1_1$ then there is some $\alpha < \omega_1$ s.t.

$$A \subseteq \bigcup_{\beta<\alpha} WO_\beta .$$

Let Γ be a pointclass. We say that a set $A \subseteq \omega^\omega$ is <u>Γ-hard</u> if for every $C \in \Gamma$ there is a cts function f s.t. $C = f^{-1}[A]$.

It is called <u>Γ-complete</u> if it is Γ-hard and itself in Γ.

<u>Lemma</u> If Γ and Δ are <u>boldface</u> pointclasses s.t. $\boxed{\Gamma \setminus \Delta \neq \emptyset}$ and there is set A which is Γ-hard. Then $A \notin \Delta$.



This is the case if $\Delta := \check{\Gamma}$ and Γ has a universal set.

<u>Pf.</u> Suppose that A is Γ-hard and that $B \in \Gamma \setminus \Delta$.

Then by def., there is f cts s.t.

$$B = f^{-1}[A].$$

If $A \in \Delta$, then since Δ is boldface, $B \in \Delta$. $\lightning$. q.e.d.

Theorem WO, WO* and WF are all $\boldsymbol{\Pi}^1_1$-hard (and thus $\boldsymbol{\Pi}^1_1$-complete). T 5.28 T 5.30 Andretta

PROOF OF BL from the Theorem.

BL: If $A \subseteq WO$ is $\boldsymbol{\Sigma}^1_1$ then there is $\alpha < \omega_1$

(*) s.t. $A \subseteq WO_{<\alpha} = \bigcup_{\beta<\alpha} WO_\beta$.

Negation of (*):

$\exists A \subseteq WO$ $\boldsymbol{\Sigma}^1_1$ s.t. $\forall \alpha < \omega_1$ $\exists y \in A$ $\|y\| > \alpha$.

Assume this and obtain a contradiction.

$$W := \{x \,;\, \underbrace{\underbrace{\exists y}_{\exists}\underbrace{\exists f}_{\exists}(\underbrace{\overset{\boldsymbol{\Sigma}^1_1}{y \in A} \wedge \underbrace{f \text{ is an inj. o.p. map from } (\mathbb{N}, E_x) \text{ into } (\mathbb{N}, E_y)}_{\text{Borel}}}_{\boldsymbol{\Sigma}^1_1})}_{\boldsymbol{\Sigma}^1_1}\}$$

$W := \{x;\ \exists y \exists f\ (y \in A \wedge f$ is an o.p. inj. from $(\mathbb{N}, E_x)$ into $(\mathbb{N}, E_y))\}$

is $\underset{\sim}{\Sigma}^1_1$

<u>Claim</u>: $W = WO$.

[That's a contradiction to the Lemma assuming $\underset{\sim}{\Pi}^1_1$-hardness of WO.]

Suppose $x \in W$. Then there are y, f as in definition. But $y \in A \subseteq WO$

So $(\mathbb{N}, E_y)$ is a <u>wellorder</u>.

But then $(\mathbb{N}, E_x)$ is a <u>wellorder</u>

so $x \in WO$.

Suppose $x \in WO$. And define $\alpha := \|x\|$.

Then by assumption $(\neg BL)$ get <u>$y \in A$</u> s.t. $\|y\| > \alpha = \|x\|$.

But then there is <u>some f</u>: $\mathbb{N} \longrightarrow \mathbb{N}$ s.t. f is an o.p. inj. from $(\mathbb{N}, E_x)$ into $(\mathbb{N}, E_y)$.

$\Longrightarrow x \in W$.

q.e.d.

Proof of Theorem (case WF).

Need to show: if $P \in \underset{\sim}{\Pi}^1_1$, there there is a cts fn $f: \omega^\omega \to \omega^\omega$ s.t.

$$\boxed{x \in P \longleftrightarrow f(x) \in WF.}$$

We had seen the tree representation of Π^1_1 sets: P is $\underset{\sim}{\Pi}^1_1 \longrightarrow$ there is a tree $T \subseteq (\omega \times \omega)^{<\omega}$ s.t.

$$\boxed{x \in P \longleftrightarrow T_x \text{ is wellfdd}}$$

$(T_x, \supsetneq)$ is wellfdd

What was T_x?

$$T_x := \{ t \in \omega^{<\omega} ; (x \restriction \underline{\text{lh}(t)}, \underline{t}) \in T \}$$
$$\subseteq \omega^{<\omega}$$

Pick your favourite bijection
$$n \longmapsto s_n$$
$$\mathbb{N} \longrightarrow \omega^{<\omega}$$

equivalently $k \in S_x \wedge \ell \in S_x \wedge k <_x \ell$

$$\boxed{\begin{aligned} S_x &:= \{ n ; s_n \in T_x \} \\ k <_x \ell &:\Longleftrightarrow s_k \supsetneq s_\ell \end{aligned}}$$

$(S_x, <_x) \cong (T_x, \supsetneq)$

Code it:
$$z_x(\langle k, \ell \rangle) := \begin{cases} 1 & s_k \in T_x \wedge s_\ell \in T_x \wedge s_k \supsetneq s_\ell \\ 0 & \text{otherwise} \end{cases}$$

$$(S_x, <_x) = (\mathrm{fld}(z_x), E_{z_x})$$

$$\boxed{x \in P} \longleftrightarrow (\mathrm{fld}(z_x), E_{z_x}) \text{ is wellfounded}$$

$$\longleftrightarrow \boxed{z_x \in WF.}$$

Remains to show that

$$x \longmapsto z_x$$

is continuous.

This is the case if I only need finitely many bits of x to determine each value of

$$z_x(\langle k, l\rangle) := \begin{cases} 1 & s_k \in T_x \wedge \\ & s_l \in T_x \wedge \\ & \boxed{s_k \supsetneq s_l} \\ 0 & \text{o/w} \end{cases}$$

$s_l \in T_x \longleftrightarrow$

$(x \restriction lh(s_l), s_l) \in T$

So in order to determine $z_x(\langle k, l\rangle)$ we need

$x \restriction N$ where $N := \max\{lh(s_k), lh(s_l)\}$

So $x \longmapsto z_x$ is cts. q.e.d.