Template Exam
Set Theory

MasterMath: 1st Semester 2020/21

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PRACTICAL ISSUES.

(1) You have 180 minutes for the exam, from 14:00 to 17:00 (unless you have been granted extra time by the Examinations Board; in that case, your extra time is added at the end). Make sure that you are in a location where (a) you can work without disturbance, (b) you can avoid contact to other people, and (c) you have the technical means to connect to elo, download the exam, produce a pdf file of your answers, and submit it at the end of the exam (cf. 2, 8, & 9).

(2) At the beginning of the exam, you can download the exam questions from elo. If you have any connectivity problems, please let us know immediately by calling either **+31 20 525 6071** or **+31 15 278 4572** and we shall try to find a solution.

(3) You can then work on the exam offline for three hours until you submit your solutions (cf. 9).

(4) During this period, you **are allowed** to use books and all of the course material provided by us, as well as all of your notes. Please make sure that you have a pdf copy of the relevant chapters of Jech's book available: this will help you to quote results when needed.

(5) You are not allowed to be in contact with any other person during the exam by any means. You will be required to confirm this with your signature (cf. 7).

(6) You write the answers to the question in handwritten form, either on paper or electronically hand-written (e.g., on a tablet with a stylus).

(7) You finish writing at 17:00. After finishing, please copy the following sentence to the last page and sign it:

> *"I hereby confirm that I worked on this exam without the help of any other person."*

(8) After this, you produce a pdf file of your written solutions. If you wrote on a tablet, make sure (in advance) that you know how to save the file in pdf format. If you wrote on paper, either scan them or use some smartphone scanning software. Also in this case, make sure (in advance) that you know how to save everything in a single file in pdf format. Please include some form of photo identification in the pdf file, i.e., either your student ID card or your national ID card or passport. Check that your file is legible and complete, in particular that you did not forget a page and that you included the confirmation statement and your signature (cf. 7), and your photo ID.

(9) After checking everything, please upload the pdf file of your answers via elo. This should be done by 17:15 (at the latest, fifteen minutes after the end of the exam; if you have been granted extra time, this time changes accordingly). If you experience connectivity problems or any other technical problems at this time, please contact us immediately by calling **+31 20 525 6071** or **+31 15 278 4572**

(10) During the 60 minutes after the exam, we may check the identity of randomly selected group of students. If you are randomly selected, you will be contacted by e-mail and invited to a video Zoom call where we check your identity and the fact that the uploaded files correspond to what you have written. For this purpose, please make sure that you are available for a video call in the 60 minutes after the end of the exam.

EXAM STRUCTURE.

(1) The exam will have two parts, a mandatory Part I and an optional Part II. Part I will have two questions and Part II will have three questions.

(2) In order to pass, you must attempt both questions in Part I. Each of them is worth $3\frac{1}{2}$ points for a total of 7 points. A satisfactory answer to both questions in Part I will be sufficient to pass the course.

(3) In addition, you may choose as many of the three questions in Part II, each of which is worth 1 point. The total exam has 10 points.

PART I.

This part of the exam is mandatory. You will not be able to get a passing exam score without answering both questions in Part I. A satisfactory answer to both questions will guarantee that you get a passing exam score.

Question I.1.

On homework sheet #2, question (7), we introduced the *Zermelo natural numbers* $\mathbb{N}_{\mathrm{Z}}$ where $0_{\mathrm{Z}} = \varnothing$ and $n + 1_{\mathrm{Z}} = \{n_{\mathrm{Z}}\}$. In analogy to the Axiom of Infinity, we can consider the *Axiom of Zermelo-Infinity*, stating that there is a Zermelo inductive set:

$$\exists I(\varnothing \in I \wedge \forall x\Big(x \in I \to \{x\} \in I\Big).$$

Explain why it is possible to replace the Axiom of Infinity with the Axiom of Zermelo-Infinity in the axioms of ZF. Pay particular attention to the role of the Axiom of Replacement in your argument.

Question I.2.

Complete the following diagram of properties of an infinite cardinal κ by stating which cases can occur and giving examples of all possible cases:

	has the tree property	does not have the tree property
weakly compact	?	?
not weakly compact	?	?

Discuss each of your examples and give reasons why it is an example (cite any theorems proved in class with proper references either to Jech's book or to the lecture notes). Explain which of the examples require going beyond ZFC.

PART II.

This part of the exam is optional and not required for passing the exam. You can answer as many questions as you like. Each question is worth one point. You can only obtain a total exam score higher than 7 points if you answer questions in Part II of the exam.

In your answers, you may use all theorems proved in class without proof, provided that you state them precisely and correctly and give a reference to Jech's book (by page number or theorem number) or to the handwritten lecture notes (with lecture number and page number).

Question II.1. Let κ be a regular cardinal. We write $\mathbf{H}_\kappa := \{x \in \mathbf{V}_\kappa\,;\, |\mathrm{TC}(x)| < \kappa\}$. Show that a cardinal κ is inaccessible if and only if $\mathbf{V}_\kappa = \mathbf{H}_\kappa$.

Question II.2. Work in ZFC and prove that $\omega_1) \to (\omega_1, \omega+1)^2$, i.e., if $F : [\omega_1]^2 \to \{0,1\}$ is a colouring then there is an uncountable 0-homogeneous set or there a 1-homogeneous set of order type $\omega + 1$.

[*Hint.* Assume there is no 1-homogeneous set of order-type $\omega+1$. Choose for every α a maximal subset K_α of α such that $K_\alpha \cup \{\alpha\}$ is 1-homogeneous. Prove that $\alpha \mapsto K_\alpha$ is constant on s stationary set.]

Question II.3. Assume that there is an elementary embedding j of the universe $\mathbf{V}$ into an inner model M with critical point κ (i.e., $j(\kappa) > \kappa$ and κ is the least ordinal with that property). Assume furthermore that $\mathbf{V}_{\kappa+2} \subseteq M$. Show that κ cannot be the least measurable cardinal.