Comments on the Template Exam
Set Theory

MasterMath: 1st Semester 2020/21

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PART I.

This part of the exam is mandatory. You will not be able to pass without answering both questions in Part I. A satisfactory answer to both questions will guarantee that you pass.

This is the most important part of the exam and you should allocate the majority of the time in the exam to it. You should definitely **complete Part I before starting to work on Part II**. It is advisable not to start writing immediately, but to first structure your thoughts in order to produce a "well-structured" answer (see below).

The two questions in Part I ask you to describe either mathematical concepts or mathematical proofs in your own words. Your answer will be marked according to whether it is **correct**, **comprehensive**, and **well-structured**. An answer is *comprehensive* if all of the important mathematical ideas are discussed and explained.

An answer will be considered **good** if all three criteria are satisfied.

It will be considered **satisfactory** if it has minor deficiencies in some of the three criteria. E.g., fixable errors in definitions or arguments would be considered a minor deficiency in correctness, the omission of one among several ideas or proof steps would be considered a deficiency in comprehensivity, a general lack of structure or confused prose would be considered a deficiency in being well-structured.

It will be considered **unsatisfactory** if it has a major deficiency in either correctness or comprehensivity, e.g., a flaw in a definition that invalidates the argument, a major error in an argument, or omitting the main idea of the proof would be considered major deficiencies.

If both of your answers in Part I are marked as **satisfactory**, you are guaranteed to pass the exam. The maximum number of points to be awarded in Part I is seven (in case both of your answers are marked as **good**).

On pages 3 to 8 of this document, we provide answers to both questions of Part I in the Template Exam. The given answers would be marked as **good**.

PART II.

This part of the exam is optional and not required for passing the exam. You can answer as many questions as you like. Each question is worth one point. You can only obtain a total exam score higher than 7 points if you answer questions in Part II of the exam.

In your answers, you may use all theorems proved in class without proof, provided that you state them precisely and correctly and give a reference to Jech's book (by page number or theorem number) or to the handwritten lecture notes (with lecture number and page number).

The questions of Part II are more similar to questions in ordinary (closed book) mathematics exams: they ask you to prove a particular statement of which you have not seen a proof before. They will be marked in the usual way for mathematics exams.

Please finish Part I of the exam before starting with Part II. You should spend a substantial part of the 180 minutes in the exam on Part I. Doing all three questions of Part II in the remaining time will be challenging: rather focus on one or two of the questions. Providing a good answers to fewer questions will be better than collecting random thoughts on all three of them.

In the following, we give some information about the answers of Questions II.1, II.2, and II.3.

Question II.1. For the direction "$\Leftarrow$", we show that if κ is not a strong limit, then $\mathbf{H}_\kappa \neq \mathbf{V}_\kappa$: let $\lambda < \kappa$ such that $2^\lambda \geq \kappa$; in particular, $\mathrm{P}(\lambda) \notin \mathbf{H}_\kappa$. But $\mathrm{P}(\lambda) \in \mathbf{V}_{\lambda+2} \subseteq \mathbf{V}_\kappa$. Thus, $\mathbf{H}_\kappa \neq \mathbf{V}_\kappa$.

For the direction "$\Rightarrow$", let us assume that κ is inaccessible. Since $\mathbf{V}_\kappa = \bigcup_{\alpha<\kappa} \mathbf{V}_\alpha$, it is enough to show for all $\alpha < \kappa$ that $\mathbf{V}_\alpha \in \mathbf{H}_\kappa$. We prove this by induction on α.

Before we start with the induction, we observe that if $A \subseteq \mathbf{H}_\kappa$ and $|A| < \kappa$, then $A \in \mathbf{H}_\kappa$: if for each $a \in A$, $|\mathrm{TC}(a)| < \kappa$, then $\mathrm{TC}(A) = A \cup \bigcup_{a\in A} \mathrm{TC}(a)$ has size $< \kappa$ by regularity of κ and the assumptions. We refer to this as the *Observation*.

Now we get to the induction: clearly, $\mathbf{V}_0 \in \mathbf{H}_\kappa$. If $\mathbf{V}_\alpha \in \mathbf{H}_\kappa$, then $|\mathbf{V}_\alpha| =: \lambda < \kappa$, and so $2^\lambda = |\mathrm{P}(\mathbf{V}_\alpha)| = |\mathbf{V}_{\alpha+1}| < \kappa$ by the fact that κ is a strong limit. Thus $\mathbf{V}_{\alpha+1}$ is a subset of $\mathbf{H}_\kappa$ of cardinality $< \kappa$, thus by the *Observation* an element of $\mathbf{H}_\kappa$.

Finally, if μ is a limit ordinal and for all $\alpha < \mu$, $\mathbf{V}_\alpha \in \mathbf{H}_\kappa$. By definition, $\mathbf{V}_\mu = \bigcup_{\alpha<\mu} \mathbf{V}_\alpha \subseteq \mathbf{H}_\kappa$. Then for all $\alpha < \mu$, the cardinal $\kappa_\alpha := |\mathbf{V}_\alpha|$ is less than κ, and thus $|\mathbf{V}_\mu| \leq \sum_{\alpha<\mu} \kappa_\alpha < \kappa$ by regularity of κ. Thus, the *Observation* implies that $\mathbf{V}_\mu \in \mathbf{H}_\kappa$.

Question II.2. This is Exercise 9.3 in Jech's book (p. 121) with a very useful hint. Since some details might not be quite clear in the hint, there are some handwritten notes to be found on pp. 10 & 11 of this file.

Question II.3. As proved in class, the assumptions of the question imply that κ is a measurable cardinal (cf. the proof of Lemma 17.3 in Jech's book). This means that there is a non-trivial κ-complete ultrafilter U on κ. If $X \in U$, then $X \subseteq \kappa$, so $X \in \mathbf{V}_{\kappa+1} = \mathrm{P}(\mathbf{V}_\kappa)$; thus, $U \subseteq \mathbf{V}_{\kappa+1}$, i.e., $U \in \mathrm{P}(\mathbf{V}_{\kappa+1}) = \mathbf{V}_{\kappa+2} \subseteq M$.

As a consequence, $M \models$"there is a non-trivial κ-complete ultrafilter on κ", and thus $M \models$"κ is measurable". Since $\kappa < j(\kappa)$, this means that in M, $j(\kappa)$ is not the least measurable cardinal, i.e.

$$M \models \exists\lambda(\lambda < j(\kappa) \wedge \text{“}\lambda \text{ is measurable”}).$$

But now we can apply elementarity (since $\mathbf{V} \models \Phi(\kappa) \iff M \models \Phi(j(\kappa))$) and get

$$\mathbf{V} \models \exists\lambda(\lambda < \kappa \wedge \text{“}\lambda \text{ is measurable”})$$

which is what we needed to show.

Important remark. *Question II.1 is easy to solve with a simple google search and* Question II.2 *is an exercise in Jech's book with a hint. In the real exams, we shall aim for questions with no easily googlable answers or solutions in Jech's book.*

Question I.1

Let ZF^* be the axiom system consisting of

EXTENSIONALITY,
PAIRING,
SEPARATION,
UNION,
POWER SET,
REPLACEMENT,
REGULARITY,

and ZERMELO-INFINITY.

We claim that ZF and ZF^* are equivalent. For this, we shall argue that

$$ZF \vdash \text{Zermelo-Infinity} \quad (1)$$

and $$ZF^* \vdash \text{Infinity}. \quad (2)$$

Proof of (1):

The formula $\Phi(x,y) :\Longleftrightarrow y = \{x\}$ is functional, so we can apply the RECURSION THEOREM (WITHOUT FIXED RANGE) from Lecture III (page 7) to obtain

$$G(0) = \emptyset$$
$$G(n+1) = y \iff \Phi(G(n), y).$$

The theorem proves that G is a function, so $\mathrm{ran}(G)$ is a set. But $\mathrm{ran}(G) = \mathbb{N}_Z$ is Zermelo inductive, so the axiom of Zermelo infinity holds.

Proof of (2): As mentioned on HW sheet #2 Q7, the theory of induction and recursion works with Zermelo inductive sets precisely as in the standard theory. So, in ZF^* we obtain an analogous version of the mentioned Recursion Theorem without fixed range for $\mathbb{N}_Z$:

THEOREM Let Φ be functional and x an arbitrary set. Then there is a unique function G s.t.

$$G(0_Z) = x$$
$$G(n+1_Z) = y \iff \Phi(G(n_Z), y)$$

The formula $\Phi(x,y) :\iff y = x \cup \{x\}$ is functional.

Thus there is a function with

$$G(0_Z) = \emptyset$$

$$G(n+1_Z) = G(n_Z) \cup \{G(n_Z)\}.$$

The range of G is the set $\mathbb{N}$ which is inductive. Thus the axiom of infinity holds.

<u>Replacement</u> The proof of the Recursion Theorem without fixed range and therefore its analogue for $\mathbb{N}_Z$ uses the Axiom of Replacement (in STEP 4 of the proof on page 8 of the lecture notes for lecture III).

Question I.2

	TREE PROPERTY	NOT TREE PROPERTY
WEAKLY COMPACT	1 W.C $\Rightarrow$ T.P.	X IMPOSSIBLE
NOT WEAKLY COMPACT	2 KÖNIG'S LEMMA	3 ARONSZAJN THEOREM

Three of the four situations are possible.

1. Every weakly compact cardinal has the tree property.
   This is Lemma 9.26 (i) in Jech.

   κ weakly compact $\Longrightarrow$ κ example for 1

2. König's Lemma says that $\aleph_0$ has the tree property (Eleventh Lecture, p. 13), but $\aleph_0$ is not weakly compact (Eleventh Lecture, p. 14).

   [Note that Jech on p. 120 defines the tree property only for uncountable cardinals.]

In class, we did not prove the possible existence of uncountable non-weakly compact cardinals with the tree property, however, we know what cardinals cannot be examples. Lemma 9.26 (ii) of Jech states that an inaccessible cardinal that has the tree property must be weakly compact.
So: examples for (2) would not be inaccessible, but rather accessible cardinals (e.g., successors $\neq \aleph_1$; cf. (3) and comments on p. 7 of the Twelfth Lecture).

(3) Aronszajn's Theorem (Thm 9.16 Jech) says that there is an Aronszajn tree
$\Longrightarrow$ $\aleph_1$ does not have the tree property
Clearly, $\aleph_1$ is not weakly compact.
So $\aleph_1$ is an example for (3).

LARGE CARDINALS

(1) The construction works in ZFC + there is a weakly compact:
Obviously, this is needed!

(2) The construction works in ZFC + there is an inaccessible. It's not clear that the extra assumption is needed.

(3) The construction works in ZFC.

All relevant definitions and theorems:

① THE FAMILY $\mathcal{H}_\alpha = \{A \subseteq \alpha : A \cup \{\alpha\}$ IS 1-HOMOGENEOUS$\}$ SATISFIES THE ASSUMPTIONS OF ZORN'S LEMMA:

IF $\mathcal{A} \subseteq \mathcal{H}_\alpha$ IS A CHAIN THEN $\bigcup \mathcal{A} \in \mathcal{H}_\alpha$

- LET $\gamma, \delta \in \bigcup \mathcal{A}$ AND TAKE $A, B \in \mathcal{A}$ WITH $\gamma \in A$ AND $\delta \in B$
  THEN $A \subseteq B$ OR $B \subseteq A$, SAY $A \subseteq B$
  THEN $\gamma, \delta \in B$ SO $F(\{\gamma, \delta\}) = 1$
- IF $\gamma \in \bigcup \mathcal{A}$ AND $\gamma \in A \in \mathcal{A}$
  THEN $F(\{\gamma, \alpha\}) = 1$ BECAUSE $A \in \mathcal{A}$

② BY ZORN'S LEMMA AND THE AXIOM OF CHOICE WE OBTAIN A FUNCTION $K : \omega_1 \to \bigcup_\alpha \mathcal{H}_\alpha$ SUCH THAT $K(\alpha)$ IS A MAXIMAL ELEMENT OF $\mathcal{H}_\alpha$

③ BY OUR ASSUMPTION $K(\alpha) \cup \{\alpha\}$ HAS ORDER-TYPE LESS THAN $\omega + 1$, HENCE IT IS A FINITE SET.

④ DEFINE $f : \omega_1 \to \omega_1$ BY $f(0) = 0$ AND $f(\alpha) = \max(K(\alpha) \cup \{0\})$ IF $\alpha > 0$.
BY FODOR'S PRESSING DOWN LEMMA THE FUNCTION f IS CONSTANT ON A STATIONARY SET S, SAY WITH VALUE β

⑤ THE SET $C = [\beta + 1]^{<\omega}$ IS COUNTABLE AND $S = \bigcup_{x \in C} \{\alpha : K(\alpha) = x\}$
HENCE THERE IS AN $x \in C$ SUCH THAT $T = \{\alpha : K(\alpha) = x\}$ IS STATIONARY.

(6) LET $\gamma < \delta$ IN T

THEN $K(\delta) \cup \{\gamma\} \subseteq \delta$

AND, BY MAXIMALITY, $K(\delta) \cup \{\gamma\} \cup \{\delta\}$ IS NOT 1-HOMOGENEOUS

BECAUSE $K(\delta) = K(\gamma)$

WE KNOW THAT

– $F(\{\eta, \gamma\}) = F(\{\eta, \delta\}) = 1$ IF $\eta \in K(\delta)$

– $F(\{\eta, \xi\}) = 1$ IF $\eta, \xi \in K(\delta)$

SO THE ONLY POSSIBILITY TO SPOIL 1-HOMOGENEITY IS BY HAVING $F(\{\gamma, \delta\}) = 0$

(7) WE FIND THAT T IS UNCOUNTABLE AND 0-HOMOGENEOUS.