Set Theory VI

12 October 2020

REMINDER An ordinal κ was called a cardinal if

$\forall \lambda < \kappa \quad \lambda \not\sim \kappa \;/\; |\lambda| \neq |\kappa|$ [bijection]

or $\forall \lambda < \kappa \quad |\kappa| \not\leq |\lambda|$. [injection]

or κ is the smallest elt of the ordinals in $[\kappa]_\sim$.

OBSERVED

- If X is a set, then $\aleph(X)$ is a cardinal.
  HARTOGS ALEPH
- If X is a set of cardinals, then $\bigcup X$ is a cardinal.

ALEPH HIERARCHY:

$$\aleph_0 = \omega$$
$$\aleph_{\alpha+1} := \aleph(\aleph_\alpha)$$
$$\aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha \quad \text{for } \lambda \text{ limit}$$

All of the aleph are cardinals. We have seen that all of the cardinals are alephs.

For X $\boxed{\text{w'oble}}$, we thus have

$|X| :=$ the unique cardinal κ s.t. $\kappa \sim X$.

$$|X| = \begin{cases} n & \text{if } X \text{ finite} \\ \aleph_\alpha & \text{if } X \text{ infinite} \end{cases}$$

If α is an ordinal,

$$\omega_\alpha := \aleph_\alpha.$$

The ordinal ω_α. The cardinal $\aleph_\alpha$.

We write ω for $\aleph_0$. NOT $\underline{\omega_0}$!

REMARK Observe that $\alpha \longmapsto \aleph_\alpha$ is a NORMAL OPERATION.

By a result from lecture V, normal operations have arbitrarily large fixed points:

Cardinals κ s.t. $\aleph_\kappa = \kappa$ are called ALEPH FIXED PTS.

$$\alpha = \aleph_\alpha$$

$$\alpha_0 = \aleph_0$$
$$\alpha_1 = \aleph_{\alpha_0} = \aleph_{\aleph_0}$$
$$\alpha_2 = \aleph_{\aleph_{\aleph_0}}$$
$$\alpha_3 = \aleph_{\aleph_{\aleph_{\aleph_0}}}$$
$$\vdots$$
$$\alpha_\infty = \bigcup_{n \in \mathbb{N}} \alpha_n$$

By our proof, this is a fixed pt.

$$\alpha_\infty = \aleph_{\alpha_\infty} = \bigcup_{\beta < \alpha_\infty} \aleph_\beta = \boxed{\bigcup_{n \in \mathbb{N}} \aleph_{\alpha_n}}$$

AXIOM OF CHOICE (AC)

Axiom of Choice (AC). *Every family of nonempty sets has a choice function.*

If S is a family of sets and $\emptyset \notin S$, then a *choice function* for S is a function f on S such that

$$f(X) \in X \tag{5.1}$$

$f : S \longrightarrow \bigcup S$

for every $X \in S$.

The Axiom of Choice postulates that for every S such that $\emptyset \notin S$ there exists a function f on S that satisfies (5.1).

Choice is not immediately obvious and it has massive influence on mainstream mathematics.

"Obviousness of axioms"

Ext, Un, Pair, Pow, Sep $\longrightarrow$ "obvious"

Repl. $\longrightarrow$ EXTRINSIC REASON
"almost obvious" (after reflection)

Found.
"not obvious"
Foundation only has consequences in the foundations of maths, not in regular mainstream maths.

EXAMPLE

Theorem A countable union of ctble sets is countable.

Proof. Let C be a ctble set s.t. for all $x \in C$ x is countable. So we have surjections

Hidden in notation
$x \longmapsto \pi_x$
is a function.

$$\pi_C : \mathbb{N} \twoheadrightarrow C$$

$$\pi_x : \mathbb{N} \longrightarrow x \quad \text{for every } x \in C.$$

$$\boxed{\begin{aligned} \pi : \mathbb{N} \times \mathbb{N} &\longrightarrow \bigcup C \\ (n, m) &\longmapsto \pi_{\pi_C(n)}(m) \end{aligned}}$$

clearly a surjection.
"q.e.d."

$\mathbb{N} \sim$

In Set Theory π has to be defined by separation applied to $\mathbb{N} \times \mathbb{N} \times \bigcup C$.

$$\pi : (n,m) \longmapsto \pi_{\pi_C(n)}(m)$$

$$\Phi(n,m,z) \boxed{\longleftrightarrow \quad z = \pi_{\pi_C(n)}(m)} \text{ WANT}$$

$$\longleftrightarrow \exists x \left(\underline{\pi_C}(n) = x \wedge \underline{\pi_x}(m) = z \right)$$

STILL NOT A FMLA IN $\mathcal{L}_\in$.

If we have the $\underline{AC}$, then consider

$\{ \underline{\mathrm{Surj}(\mathbb{N}, x)} ; x \in C \cup \{C\} \}$ this is a set of non-empty sets, so it has a choice fn f by $\underline{AC}$

$$\longleftrightarrow \exists x \exists p \exists q \left(p = f(C) \wedge x = p(n) \wedge q = f(x) \wedge z = q(m) \right)$$

(REAL) q.e.d.

Remarks :

1. The use of AC is usually hidden in notation like " f_x f.e. x " which hides the ex. of a function $x \mapsto f_x$.

2. Warning signs are : CHOOSE / PICK.
   But they are not always there.

3. Just because you use AC doesn't mean that you must use AC.
   In some cases, the existence of a choice function is provable in ZF.

Examples for 3. $\longrightarrow$

Ex. • If $S = \{x\}$ with $x \neq \emptyset$.
For each $z \in x$, the following object
$$\{(x, z)\}$$
is a choice function for S.
• If all of the elements of S are sets of ordinals, then
$$\{(x, \alpha) ; \alpha = \min(x)\}$$
is a choice function for S.

AC appears in Zermelo 1908.
Zermelo uses it to justify the proof of his "Wellordering Theorem". MORE He show that AC and his WOT are equivalent.

From 1908 to ... (???), the AC was the only axiom that
2020
mathematicians liked to discuss.

Theorem in ordinary maths that require some use of AC:

ALGEBRA

- Every vector space has a basis,
- Every field has an algebraic closure.
- Every field has a transcendence base over its prime field.

ANALYSIS

- Eq. between ε-δ continuity and "f is cts iff $\forall x_n \to x$ $f(x_n) \to f(x)$"
- Hahn-Banach
- Existence of non-LM set.

<u>Aside</u>. AC is equivalent to
"if I is an index set and X_i is a non-empty set for each $i \in I$, then

$$\prod_{i \in I} X_i \neq \emptyset \text{"}.$$

$$\| $$

$$\{ f \,;\, \mathrm{dom}(f) = I \text{ and } \forall i \in I \;\; f(i) \in X_i \}$$

These are precise the choice fns for the family $X = \{ X_i \,;\, i \in I \}$.

Theorem (Zermelo's Wellordering Theorem)
ZWOT

AC $\Longrightarrow$ every set can be wellordered.

Proof. Let X be an arbitrary set. It's enough to show that $X \sim \alpha$ where α is an ordinal.

Consider $\eta := \aleph(X)$. The Hartogs aleph: η does not inject into X.

Let f be a choice function for $P(X)\setminus\{\emptyset\}$.

Define by recursion

$$F(\alpha) := \begin{cases} f(X\setminus \mathrm{ran}(F\restriction\alpha)) & \text{if } X\setminus\mathrm{ran}(F\restriction\alpha) \neq \emptyset \\ \text{STOP} & \text{o/w.} \end{cases}$$

1. $\gamma := \{\alpha ; F(\alpha) \neq \text{STOP}\}$ is an initial segment of η.

2. On γ, F is injective:

$F \upharpoonright \gamma : \gamma \longrightarrow X$ is an injection.

$\Longrightarrow \gamma \neq \eta \Longrightarrow \gamma \in \eta$.

3. $F(\gamma) = \text{STOP}$.

$\Longrightarrow \operatorname{ran}(F \upharpoonright \gamma) = X$

surjection

$\Downarrow$

$\Longrightarrow F \upharpoonright \gamma : \gamma \longrightarrow X$ is a bijection.

$\Longrightarrow X$ is w'oble.

q.e.d.

Remark. This proof uses AC, but that in itself is not proof that AC is needed.

"Needing" an axiom means that AC can be proved from assuming the conclusion.

In this case by proving

$$\boxed{ZF + WOT \text{ implies } AC.}$$

Note that this only shows that AC is "needed" if we know that AC is not a theorem of ZF. Indeed, ZF does not prove AC, but this a highly nontrivial result & beyond the scope of this course.

Theorem ZF + WOT proves AC.

Proof. Let X be a set of nonempty sets. We need choice fn: $f : X \longrightarrow \boxed{\bigcup X}$

s.t. $\forall z \in X \; f(z) \in z$.

Wellorder $\bigcup X$ by WOT: Let R be a relation on $\bigcup X$, i.e., $R \subseteq \bigcup X \times \bigcup X$ s.t. $(\bigcup X, R)$ is a wellorder.

$$f(z) := a \;:\Longleftrightarrow\; \boxed{a \in z \wedge \forall b \, (b \in z \longrightarrow aRb \text{ or } a = b)}$$

$f(z) =$ R-minimal elt. of z

q.e.d.

<u>Back to cardinals</u>. In ZFC it makes sense to define $|X| :=$ the unique cardinal κ s.t. $X \sim \kappa$.

$$|X| = |Y| \iff \exists \pi : X \longrightarrow Y \text{ bijective}$$
$$\iff \exists \pi : \kappa \longrightarrow \lambda \text{ bij. } \& \ \kappa = |X| \ \& \ \lambda = |Y|$$

CP (COMPARISON PRINCIPLE)
If X & Y are sets, then $|X| \leq |Y|$ or $|Y| \leq |X|$.

<u>Thm</u> AC $\Longrightarrow$ CP.

<u>Pf.</u> By WOT, $X \sim \alpha$, $Y \sim \beta$ ordinals and thus either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. q.e.d.

Thm CP $\Longrightarrow$ AC.

Pf. By the previous thm, it's enough to show that CP $\Longrightarrow$ WOT.

Show that X is w'oble.

Compare X with $\aleph(X)$. By CP, we have:

$$|X| \leq |\aleph(X)| \quad \text{or} \quad \cancel{|\aleph(X)| \leq |X|}$$

Contradicts the def. of $\aleph(X)$

Thus X injects into an ordinal and so X is w'oble. q.e.d.

$\boxed{ZFC} := \underline{ZF + AC}$.

From now on, we work in ZFC!

Let's study these CARDINALS:

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \dots$ $\qquad$ $\aleph_0 \quad \boxed{\aleph_1} \quad \aleph_2 \quad \aleph_3 \quad \aleph_4 \quad \aleph_5 \quad \dots \quad \boxed{\aleph_\omega} \quad \aleph_{\omega+1} \quad \aleph_{\omega+2} \quad \dots \quad \boxed{\aleph_{\omega\cdot 2}} \quad \aleph_{\omega\cdot 2+1} \quad \dots$

$\mathbb{N}$ $\qquad$ ALEPHS

has an immediate predecessor

$\aleph_1 = \aleph(\aleph_0)$

SUCCESSOR CARDINALS

don't have immediate predecessor

LIMIT CARDINALS

$$\aleph_0 := \omega$$

$$\aleph_{\alpha+1} := \aleph(\aleph_\alpha)$$

$$\aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha$$

Def. A cardinal $\aleph_\alpha$ is called SUCCESSOR CARDINAL (LIMIT CARDINAL) iff α is a successor ordinal (limit ordinal).

NOTE OF CAUTION

Every infinite cardinal (including successor cards) is a limit ordinal.

[Why? Suppose $\kappa = \alpha + 1$.

Since α is infinite $\omega \leq \alpha$.

By subtraction $\alpha = \omega + \gamma$

$$\alpha = \underline{\omega} + \gamma = \underline{1 + \omega} + \gamma$$

$$\alpha + 1 = \boxed{\omega + \gamma} + 1$$

So $\alpha \sim \alpha + 1$ are in bijection

Thus κ is not a cardinal.]

The first limit cardinals we see : $\aleph_\omega, \aleph_{\omega+\omega}, \aleph_{\omega+\omega+\omega}$
are all unions of the form

$$\aleph_\omega = \boxed{\bigcup_{n<\omega} \aleph_n}$$

These are all countable unions of smaller cardinals.

$$\aleph_{\underline{\omega+\omega}} = \bigcup_{\xi<\omega+\omega} \aleph_\xi = \boxed{\bigcup_{n\in\omega} \aleph_{\omega+n}}$$

$$\aleph_{\underline{\omega+\omega+\omega}} = \bigcup_{\xi<\omega\cdot 3} \aleph_\xi = \boxed{\bigcup_{n\in\omega} \aleph_{\omega\cdot 2+n}}$$

If $\underline{\alpha < \omega_1}$ is any limit ordinal,

then $\omega_\alpha = \boxed{\bigcup_{\beta<\alpha} \omega_\beta}$ is a ctble union of smaller objects.

What about the successors?

If α limit ordinal

$X \subseteq \alpha$ COFINAL / UNBOUNDED in α

if $\forall \beta < \alpha \, \exists \gamma \in X \, (\beta < \gamma)$.

If $f : \delta \longrightarrow \alpha$ any function,

f is COFINAL if $\operatorname{ran}(f) \subseteq \alpha$ is cofinal.

$\beta \longrightarrow \alpha$

$\xi \longmapsto \alpha_\xi$

Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

$$\text{cf}\,\alpha = \text{the least limit ordinal } \beta \text{ such that there is an increasing } \beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha.$$

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$\operatorname{cf} \alpha := \min \{ \beta ;$ β limit ordinal s.t. there is an increasing & cofinal fn $f : \beta \longrightarrow \alpha \}$.

PROPERTIES

1. $\operatorname{cf} \alpha \leq \alpha$
[$\mathrm{id} : \alpha \longrightarrow \alpha$ is increasing & cofinal]

2. $\operatorname{cf} \alpha$ is a limit ordinal

(2) follows from definition.
[but it's not needed in the definition]

3. $\operatorname{cf} \operatorname{cf} \alpha = \operatorname{cf} \alpha$

(1) implies $\operatorname{cf} \operatorname{cf} \alpha \leq \operatorname{cf} \alpha$

[If $\beta := \operatorname{cf} \alpha$; $\gamma = \operatorname{cf} \beta$.

$f : \beta \longrightarrow \alpha$ incr. + $g : \gamma \longrightarrow \beta$ cof. $\Longrightarrow f \circ g : \gamma \longrightarrow \alpha$ incr. + cof.

$\Longrightarrow \operatorname{cf} \alpha \leq \operatorname{cf} \operatorname{cf} \alpha$]

④ If $f:\beta \longrightarrow \alpha$ is cofinal (not nec. increasing),
then $\mathrm{cf}\,\alpha \leq \beta$.

[Consider $B := \{\gamma \in \beta;\ f(\gamma) > f(\delta)$
f.a. $\delta < \gamma\}$

Then $\underline{\underline{f \upharpoonright B : B \longrightarrow \alpha}}$ is increasing
check that it is still cofinal !!

By Representation Thm find β^* s.t.
$\underline{(\beta^*, \in) \cong (B, \in)}$ $\quad \underline{\beta^* \leq \beta}$

Combining the iso with $f \upharpoonright B$ gives a cof. + incr. fn
from β^* into α. $\Longrightarrow \mathrm{cf}\,\alpha \leq \beta^* \leq \beta$.]

(5) $\underline{\mathrm{cf}\,\alpha}$ is always a cardinal.

[Why? Let $\boxed{\lambda \leq \mathrm{cf}\,\alpha}$ and $\lambda \sim \mathrm{cf}\,\alpha$

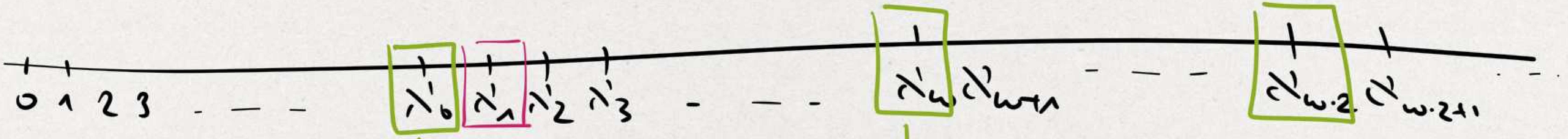
$\pi : \lambda \longrightarrow \mathrm{cf}\,\alpha$ bijection.

Let $f : \mathrm{cf}\,\alpha \longrightarrow \alpha$ be cofinal + incr.

Then $f \circ \pi : \lambda \longrightarrow \alpha$ is cofinal, by (4)

we get $\boxed{\mathrm{cf}\,\alpha \leq \lambda}$

$\lambda = \mathrm{cf}\,\alpha. \implies \mathrm{cf}\,\alpha$ is a cardinal.



$\mathrm{cf}\,\aleph'_0 = \aleph'_0$

$\aleph'_\omega = \bigcup_{n\in\omega} \aleph'_n$

$n \longmapsto \aleph'_n$

is increasing + cofinal

$\mathrm{cf}\,\aleph'_\omega \leq \aleph'_0$.

= follows from:

$\{\alpha_0, \dots, \alpha_n\}$ find $k_0, \dots, k_n$ s.t.

$\alpha_0 \in \aleph'_{k_0}$, $\alpha_i \in \aleph'_{k_i}$.

$N := \max(k_0, \dots, k_n)$

$\{\alpha_0, \dots, \alpha_n\} \subseteq \aleph'_{N+1}$.

$\mathrm{cf}\,\aleph'_\omega = \aleph'_0$

$n \longmapsto \aleph'_{\omega+n}$

is increasing & cofinal

$\mathrm{cf}\,\aleph'_{\omega\cdot 2} \leq \aleph'_0$

= follows similarly.

Def. A cardinal κ is called REGULAR if $\mathrm{cf}\,\kappa = \kappa$. Otherwise it's called SINGULAR.

$$\left[\mathrm{cf}\,\kappa < \kappa \iff \exists f : \mathrm{cf}\,\kappa \longrightarrow \kappa \text{ cofinal + incr.}\right]$$

Theorem AC $\Longrightarrow$ every successor cardinal is regular

Pf. Let $\aleph_{\alpha+1} = \aleph(\underline{\underline{\aleph_\alpha}})$ be a successor cardinal.

If $\gamma < \aleph_{\alpha+1}$, then there is a surj. $\boxed{\pi_\gamma : \aleph_\alpha \longrightarrow \gamma.}$

AC: $\mathrm{Surj}(\aleph_\alpha, \gamma) \neq \emptyset$

Choice fn π: $\{\mathrm{Surj}(\aleph_\alpha, \gamma) ; \gamma < \aleph_{\alpha+1}\}$

[Think why this is w.l.o.g.]

Assume that $\boxed{f : \underline{\underline{\aleph_\alpha}} \longrightarrow \aleph_{\alpha+1}}$ is cofinal towards a contradiction.

$f : \aleph_\alpha \longrightarrow \aleph_{\alpha+1}$ cofinal

$\forall \gamma < \aleph_{\alpha+1}$ $\quad \pi_\gamma : \aleph_\alpha \twoheadrightarrow \gamma$ surjective

$$g : \aleph_\alpha \times \aleph_\alpha \longrightarrow \aleph_{\alpha+1}$$
$$(\gamma, \delta) \longmapsto \pi_{f(\gamma)}(\delta)$$

Check that g is a surjection from $\aleph_\alpha \times \aleph_\alpha$ onto $\aleph_{\alpha+1}$.

This contradicts HESSENBERG's THM:

$\aleph_\alpha \times \aleph_\alpha \sim \aleph_\alpha$. [Together get surj. from $\aleph_\alpha$ onto $\aleph_{\alpha+1}$]

Contradiction! q.e.d.

One last comment about cofinality:

If λ is a limit ordinal

$$\operatorname{cf} \aleph_\lambda = \operatorname{cf} \lambda.$$

[Obviously $\operatorname{cf} \aleph_\lambda \leq \operatorname{cf} \lambda$: $g : \operatorname{cf}\lambda \to \lambda$ is cofinal

$\alpha \mapsto \aleph_{g(\alpha)}$ is cofinal in $\aleph_\lambda$.]

[Think about the other direction!]

In particular: $\underline{\operatorname{cf} \aleph_{\aleph_1}} = \operatorname{cf} \aleph_1 = \underline{\aleph_1}$.

$\aleph_{\aleph_1}$ is singular.

CARDINAL ARITHMETIC

We had arithmetic on ordinals, so how we do this on cardinals.

$$\kappa, \lambda \text{ cardinals } (\Longrightarrow \text{ ordinals})$$

$$\text{so } \kappa + \lambda \text{ (ordinal arithmetic).}$$

But $\kappa + \lambda$ is not always a cardinal:

$$\kappa = \lambda = \omega, \quad \kappa + \lambda = \omega + \omega. \text{ But } |\omega + \omega| = \aleph_0$$

Easy fix:

$$\kappa \boxplus \lambda := |\kappa + \lambda| \quad \leftarrow \text{ord. arithmetic.}$$

$$\kappa \boxtimes \lambda := |\kappa \cdot \lambda|$$

These operations coincide with $+$ and $\cdot$ on $\mathbb{N}$.

HESSENBERG's THM (GI#5) shows that

$$\kappa \boxplus \lambda = \kappa \boxtimes \lambda = \max\{\kappa, \lambda\}$$

if $\aleph_0 \leq \kappa, \lambda$

NOTE OF CAUTION. Authors use $+, \cdot$ for these operations.

E.g., $\aleph_0 \cdot \aleph_0 = \aleph_0$

$\omega \cdot \omega = \omega^2 \neq \omega$

card. mult.

ordinal mult.

What about exponentiation? What about $\operatorname{Exp}(\kappa, \lambda) := |\kappa^\lambda|$?

There is a much more interesting exponentiation operation:

$$\mathrm{Func}(Y,X) = X^Y := \{ f \,;\, \mathrm{dom}(f) = Y \wedge \mathrm{ran}(f) \subseteq X \}$$

$$\underline{\underline{\kappa^\lambda}} := |\mathrm{Func}(\lambda, \kappa)|.$$

CARDINAL EXPONENTIATION

$$\boxed{2^{\aleph_0}} = |\mathrm{Func}(\mathbb{N}, 2)| = |\mathbb{R}| > \aleph_0$$

This is completely different from ordinal exp:

$$\boxed{2^\omega} = \bigcup_{n \in \omega} 2^n = \omega.$$

Cantor's Thm For all κ,

$$2^\kappa > \kappa.$$