Set Theory

FIFTH LECTURE

5 October 2020

(0) *Ordinal subtraction.* Read and understand the following argument in detail:

If $\beta \leq \alpha$, then there is a unique ordinal γ such that $\alpha = \beta + \gamma$.

(1) *Ordinal Division (with remainder).* Let $0 < \beta < \alpha$. Show that there are unique γ and ϱ such that $\alpha = \beta \cdot \gamma + \varrho$ and $\varrho < \beta$.

(3) *Ordinal Logarithm (with remainders).* Let $1 < \beta < \alpha$. Show that there are unique γ, ϱ_0, and ϱ_1 such that

(a) $\alpha = \beta^\gamma \cdot \varrho_0 + \varrho_1$,

(b) $\varrho_0 < \beta$, and

(c) $\varrho_1 < \beta^\gamma$.

(4) Let γ be an ordinal. A finite sequence $(\gamma_0, ..., \gamma_n)$ with $\gamma_0 \geq \gamma_1 \geq ... \geq \gamma_n$ is called a *Cantor Normal Form of* γ if

$$\gamma = \omega^{\gamma_0} + \cdots + \omega^{\gamma_n}.$$

Prove that every ordina' $\gamma > 0$ has a unique Cantor Normal Form.

GROUP INTERACTION #4

(12) Prove the following properties of ordinal arithmetic (in the following, α, β, and γ are ordinals):

(a) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$. ASSOCIATIVITY

(b) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. DISTRIBUTIVITY

(c) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

(d) $\alpha^{\beta \cdot \gamma} = (\alpha^\beta)^\gamma$.

] Exponential rules

(13) Prove the following monotonicity properties of ordinal arithmetic (in the following, α, β, and γ are ordinals):

(a) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.

(b) If $\alpha < \beta$, then $\underline{\gamma + \alpha} < \underline{\gamma + \beta}$.

(c) If $\alpha \leq \beta$, then $\underline{\alpha \cdot \gamma} \leq \underline{\beta \cdot \gamma}$.

(d) If $\alpha < \beta$ and $\gamma \neq 0$, then $\underline{\gamma \cdot \alpha} < \underline{\gamma \cdot \beta}$.

HOMEWORK #4

MORE ON ORDINALS

(1) Asymmetry of definitions implies:

If β is a limit : $\alpha + \beta$ limit

—— " —— , $\alpha \neq 0$: $\alpha \cdot \beta$ limit

—— " —— $\alpha \neq 0, 1$: α^β limit

This is behind many of the questions:

If α successor, β limit: for all γ $\gamma + \alpha \neq \beta$

(2) CNF:

$$\alpha = \omega^{\gamma_0} + \ldots + \omega^{\gamma_n} \qquad \gamma_0 \geq \ldots \geq \gamma_n$$

$$= \underbrace{\omega^{\gamma_0} \cdot k_0}_{} + \ldots + \underbrace{\omega^{\gamma_n} \cdot k_n}_{} \qquad k_i \in \mathbb{N}$$

$$\gamma_0 > \ldots > \gamma_n$$

block of terms with same exponent

For every γ, ω^{γ} is a limit unless $\gamma = 0$. $[\omega^0 = 1]$

$[\omega^{\lambda}$ is a limit if λ is limit$]$

$[\omega^{\delta+1} = \omega^{\delta} \cdot \omega$ is a limit$]$

So $\alpha = \omega^{\gamma_0} + \ldots + \omega^{\gamma_n} + \underbrace{1 + 1 + \ldots + 1}_{k_{n+1} \text{ times}}$

Thus α can be written uniquely as $\lambda + n$ where λ is limit & $n \in \mathbb{N}$.

So α is EVEN if n is even where $\alpha = \lambda + n$. [Same: ODD]

③ Applications of CNF.

Claim: If $\gamma < \omega^2$, then $\gamma + \omega^2 = \omega^2$.

[By CNF $\gamma = \omega^1 \cdot k_1 + \omega^0 \cdot k_0 = \omega \cdot k_1 + k_0$
$< \omega \cdot k_1 + \omega$
$= \omega \cdot (k_1 + 1)$

$$\begin{aligned}
\gamma + \omega^2 &\leq \omega \cdot (k_1 + 1) + \omega^2 \\
&= \omega \cdot (k_1 + 1) + \omega \cdot \omega \\
&= \omega \cdot (\cancel{k_1 + 1} + \omega) \\
&= \omega^2 \leq \gamma + \omega^2 .
\end{aligned}$$

[Distr. to the right]

④ In general, $\beta \neq 0$, if $\gamma < \omega^{\beta}$, then $\gamma + \omega^{\beta} = \omega^{\beta}$.

⑤ By exponentiation laws: $\gamma < \omega^{\beta}$

$$\omega^{\gamma} \cdot \omega^{(\omega^{\beta})} = \omega^{(\gamma + \omega^{\beta})} = \omega^{(\omega^{\beta})}.$$

[④, ⑤ are related to the question on gamma & delta numbers on GI #4]

⑥ Calculemus

$$\underline{(\omega+1)\cdot 2} = \omega + \cancel{1} + \omega + 1$$
$$= \omega + \omega + 1 = \underline{\omega \cdot 2 + 1}$$

NOT $\omega \cdot 2 + 2$!!!

$$\underline{2 \cdot (\omega + 1)} \overset{D}{=} 2 \cdot \omega + 2 \cdot 1$$
$$= \underline{\omega + 2}$$

⑦ $n \neq 0$ $(\omega^\alpha + \gamma) \cdot n = \underbrace{\omega^\alpha + \cancel{\gamma} + \omega^\alpha + \cancel{\gamma} + \ldots + \omega^\alpha + \gamma}_{n \text{ times}}$

where $\gamma < \omega^\alpha$

$= \omega^\alpha \cdot n + \gamma < \omega^\alpha (n+1)$

⑧ $(\omega^\alpha + \gamma) \cdot \omega = \bigcup_{n \in \mathbb{N}} (\omega^\alpha + \gamma) \cdot n = \bigcup_{n \in \mathbb{N}} \omega^\alpha \cdot (n+1)$

$= \omega^\alpha \cdot \omega$

(9) $(\omega^3 + \omega^2 \cdot 2 + 1)(\omega^2 + \omega \cdot 3 + 2)$

$$= \underbrace{(\omega^3 + \omega^2 \cdot 2 + 1)\omega^2}_{(\omega^3 + \omega^2 \cdot 2 + 1) \cdot \omega \cdot \omega} + (\omega^3 + \omega^2 \cdot 2 + 1)\omega \cdot 3 + (\omega^3 + \omega^2 \cdot 2 + 1) \cdot 2$$

[Distr. from right]

disappears by (8)

$$\omega^3 + \cancel{\omega^2 \cdot 2 + 1} + \omega^3 + \omega^2 \cdot 2 + 1$$

$$= \omega^5 + \omega^4 \cdot 3 + \omega^3 \cdot 2 + \omega^2 \cdot 2 + 1$$

Usual polynomial multiplication:

$$(x^3 + 2x^2 + 1)(x^2 + 3x + 2) = x^5 + 5x^4 + 8x^3 + 5x^2 + 3x + 2$$

(10) SYNTHETIC DEFINITIONS OF PLUS & TIMES

($\neq$ INDUCTIVE)

If α, β are ordinals, then $(\alpha, \in), (\beta, \in)$ are well-orders, $(\alpha, \in) \oplus (\beta, \in)$ is a wellorder.

By Representation Thm, there is a unique γ s.t.

$$(\gamma, \in) \cong (\alpha, \in) \oplus (\beta, \in)$$

Define $\alpha \oplus \beta := \gamma$.

$$\boxed{\text{Theorem For all } \alpha, \beta \quad \alpha \oplus \beta = \alpha + \beta.}$$

Proof of Theorem $\alpha \oplus \beta = \alpha + \beta$ by induction on β.

$$\underbrace{(\alpha,\in) \oplus (\beta,\in)}$$

$$M = \boxed{\alpha \times \{0\} \cup (\beta \times \{1\})}$$

$(\gamma,i) < (\delta,j)$
$:\Longleftrightarrow (i=j \text{ and } \gamma<\delta)$ or
$(i=0 \text{ and } j=1)$

We show a strengthened statement by induction:

for α,β the map
$$(\gamma,0) \longmapsto \gamma$$
$$(\delta,1) \longmapsto \alpha+\delta$$
is an isomorphism between M and $\alpha+\beta$.

$\beta=0$ $M = \alpha \times \{0\}$

$\beta=\gamma+1$ and the map for α,γ is an isomorphism.
The map for α,β only has one extra entry:
$$(\gamma,1) \longmapsto \alpha+\gamma$$
So clearly an isom.

β is limit; by ind. we have that for $\gamma < \beta$

$\pi_\gamma : \alpha \times \{0\} \cup \gamma \times \{1\} \longrightarrow \alpha + \gamma$

is an isomorphism

Of course, if $\gamma < \gamma'$ then $\pi_\gamma \subseteq \pi_{\gamma'}$.

$$\pi_\beta = \bigcup_{\gamma < \beta} \pi_\gamma$$

Check that π_β as a union of isomorphisms is an isomorphism.

q.e.d.

Synthetic definition of TIMES:

$(\alpha, \in) \otimes (\beta, \in)$ is a wellorder : $\alpha \otimes \beta = \gamma$ if γ is the unique ordinal s.t.

$(\gamma, \in) \cong (\alpha, \in) \otimes (\beta, \in)$.

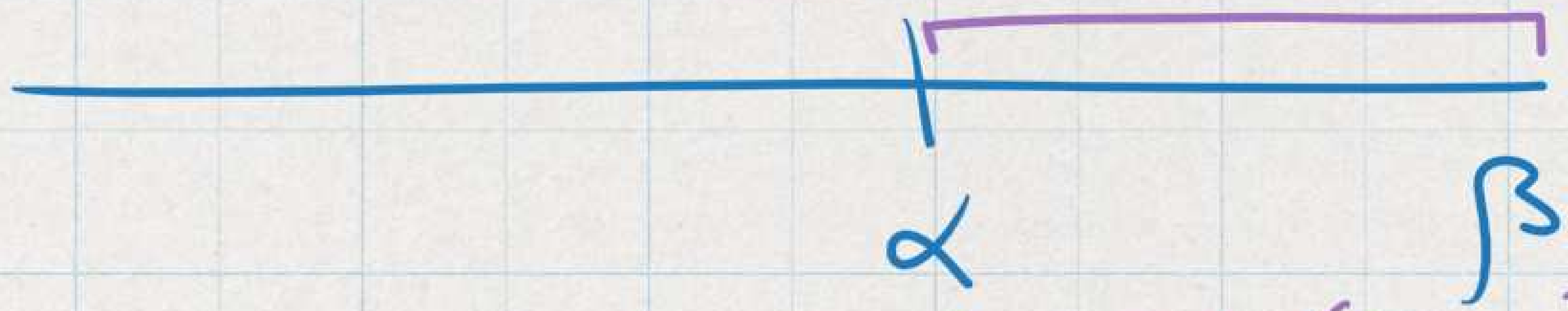
THEOREM $\alpha \cdot \beta = \beta \otimes \alpha$.

HW #5 contains a "synthetic definition of exponentiation".

(11) <u>Application</u>

Proof of the subtraction theorem. [(0) on GI #4]

$\alpha \leq \beta$ then there is γ s.t. $\alpha + \gamma = \beta$.



$(\beta \setminus \alpha, \in)$ is a well order

find γ s.t. $(\gamma, \in) \underset{\pi}{\cong} (\beta \setminus \alpha, \in)$

$(\alpha, \in) \oplus (\gamma, \in) \cong (\beta, \in)$

OBVIOUS ISO: $\delta \in \alpha \longmapsto \delta$

$\eta \in \gamma \longmapsto \pi(\eta) \in \beta \setminus \alpha$.

(12) GI # 4 asked about α s.t. $\boxed{\omega^\alpha = \alpha.}$ ← continuous increasing

An assignment of ordinals to ordinals:

$$\alpha \longmapsto F(\alpha)$$

[Def. 2.17 Jech]

is called **NORMAL** (normal sequence)

if $\forall \alpha, \beta \quad \alpha < \beta \longrightarrow F(\alpha) < F(\beta)$ INCREASING

and $\forall \lambda$ limit $\quad F(\lambda) = \bigcup_{\gamma<\lambda} F(\gamma)$ CONTINUOUS

<u>Theorem</u> If F is a normal sequence, then for each α, there is a $\beta \geq \alpha$ s.t. $F(\beta) = \beta$.

<u>FIXED POINT</u>

Proof. Fix α. If $\alpha = F(\alpha)$ done.
[Increasing implies $\alpha \leq F(\alpha)$. (since the ordinals are wellordered)]

Otherwise Then $\alpha < F(\alpha)$.

$$\alpha < F(\alpha) < F(F(\alpha)) < F(F(F(\alpha))) < \dots \longrightarrow \lambda$$

$$\alpha_0 := \alpha$$
$$\alpha_{n+1} := F(\alpha_n)$$

strictly increasing seq.; So $\{\alpha_n ; n \in \mathbb{N}\}$ has no largest element; $\lambda := \bigcup_{n\in\mathbb{N}} \alpha_n$ is a limit ordinal

Claim $F(\lambda) = \lambda$

$$F(\lambda) = \bigcup_{\gamma < \lambda} F(\gamma) = \bigcup_{n\in\mathbb{N}} F(\alpha_n) = \bigcup_{n\in\mathbb{N}} \alpha_{n+1} = \lambda.$$

CARDINALS

Notion of "having the same size":

$X \sim Y :\Longleftrightarrow$ there is a bijection from X to Y.

Often we write $|X| = |Y|$

"the cardinality of X equals the cardinality of Y"

This would require some assignment of $|X|$ to X s.t.

$$X \sim Y \Longleftrightarrow |X| = |Y|.$$

[looking for "canonical representatives" in the $\sim$-equivalence classes]

For now, we just think of $|X| = |Y|$ as eq. to $X \sim Y$.

$|X| = |Y|$

<u>Jech</u>: $|X| \leq |Y| :\Longleftrightarrow$ there is an injection from X to Y

$|X| < |Y| :\Longleftrightarrow |X| \leq |Y|$ & $X \not\sim Y$.

<u>Theorem</u> (Cantor-Schröder-Bernstein)
If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $X \sim Y$.

[Jech, Thm 3.2]
Different proof on HW#5.

<u>Lemma</u> If $|X| \leq |Y|$ and $X \neq \emptyset$,
then there is a surjection from Y to X.

<u>Pf.</u> If $f: X \longrightarrow Y$ is injection.
$f^{-1}: \mathrm{ran}(f) \longrightarrow X$ is a surjection.
$\mathrm{ran}(f) \subseteq Y$

Pick $x_0 \in X$ and

$$g: y \longmapsto \begin{cases} f^{-1}(y) & \text{if } y \in \mathrm{ran}(f) \\ x_0 & \text{o/w.} \end{cases}$$

q.e.d.

Remark You might think that for $X \neq \emptyset$ the following are eq.:

(i) $|X| \leq |Y|$

(ii) there is a surj. from Y to X.

But the direction (ii) $\Longrightarrow$ (i) needs an axiom that we do not have yet.

Remember X was finite if there is $n \in \mathbb{N}$ s.t. $X \sim n$.

Here the canonical representative $|X| := n$.

[X, Y finite, $X \sim Y \iff |X| = |Y|$]

in this sense

Observe If X is finite, $Y \subsetneqq X$, then $X \not\sim Y$.

This is not in general true for infinite sets:

$E \subsetneqq \mathbb{N}$ $\qquad E := \{2n;\, n \in \mathbb{N}\}$

$n \longmapsto 2n$ is a bijection between $\mathbb{N}$ & E

$O := \{2n+1;\, n \in \mathbb{N}\}$

$n \longmapsto 2n+1$ is bij. between $\mathbb{N}$ & O.

Def. A set X is called Dedekind-finite if and only if for all $Y \subsetneqq X$, $X \not\sim Y$.

By the above, every finite set is D-finite.

Is every D-finite set finite?

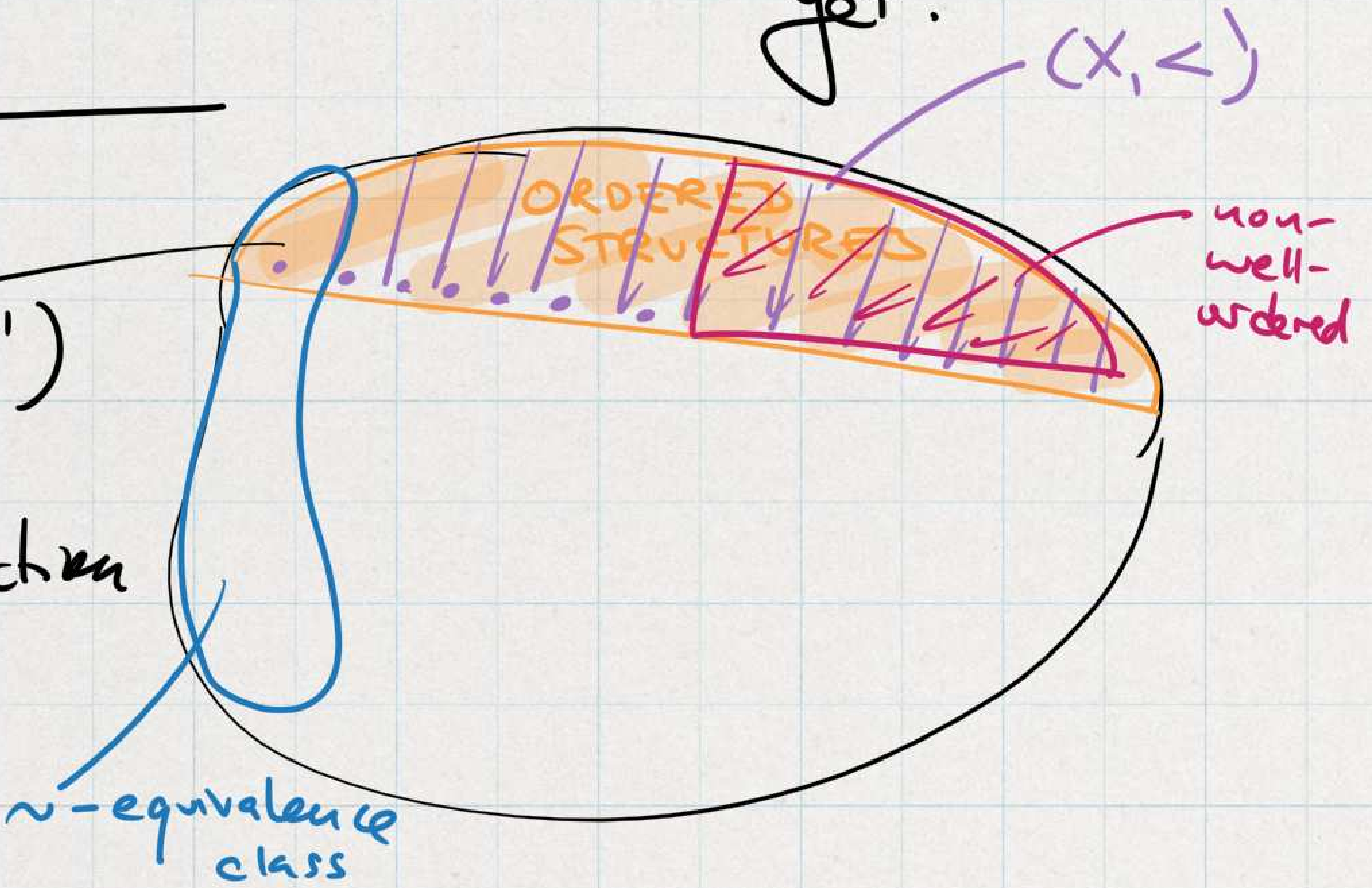
A: This needs an additional axiom that we didn't have yet.

$(X,R) \cong (Y,R')$

then $f: X \longrightarrow Y$

orderpreserving bijection

$\Longrightarrow X \sim Y$



Definition A set X is called WELLORDERABLE (W'OBLE)

if there is some $R \subseteq X \times X$ s.t. (X, R) is a wellorder.

Examples

- α is an ordinal, then $\in$ is a relation on α that makes $(\alpha, \in)$ a wellorder.
- $\mathbb{N}$ are w'oble
- $\mathbb{Z}$ are w'oble : $(\mathbb{Z}, <)$ not a wellorder, there is a bijection $\pi : \mathbb{Z} \longrightarrow \mathbb{N}$, order $\mathbb{Z}$ by the bijection $z R z' :\Longleftrightarrow \pi(z) < \pi(z')$

$$(\mathbb{Z}, R) \cong (\mathbb{N}, <)$$

In general, if X w'oble, $Y \sim X$ then Y w'oble.

Lemma For every set X the foll. are eq.:

(1) X is wellorderable

(2) there is an ordinal α s.t. X injects into α

(3) there is an ordinal α s.t. $X \sim \alpha$.

Pf. (3) $\Longrightarrow$ (2) ✓

(1) $\Longrightarrow$ (3) (X, R) is wellorder $\xRightarrow[\text{wellord.}]{\text{Repr. thm}}$ $\exists \alpha\ (X,R) \cong (\alpha, \in)$

$\Longrightarrow X \sim \alpha$.

(2) $\Longrightarrow$ (1) Let $f: X \longrightarrow \alpha$ be an injection.

$Y := \operatorname{ran}(f) \subseteq \alpha$ $\quad$ $xRx' :\Longleftrightarrow f(x) \in f(x')$

$f: (X,R) \cong \underline{\underline{(Y, \in)}}$ $\quad$ this is a wellorder as a subset of a wellord.

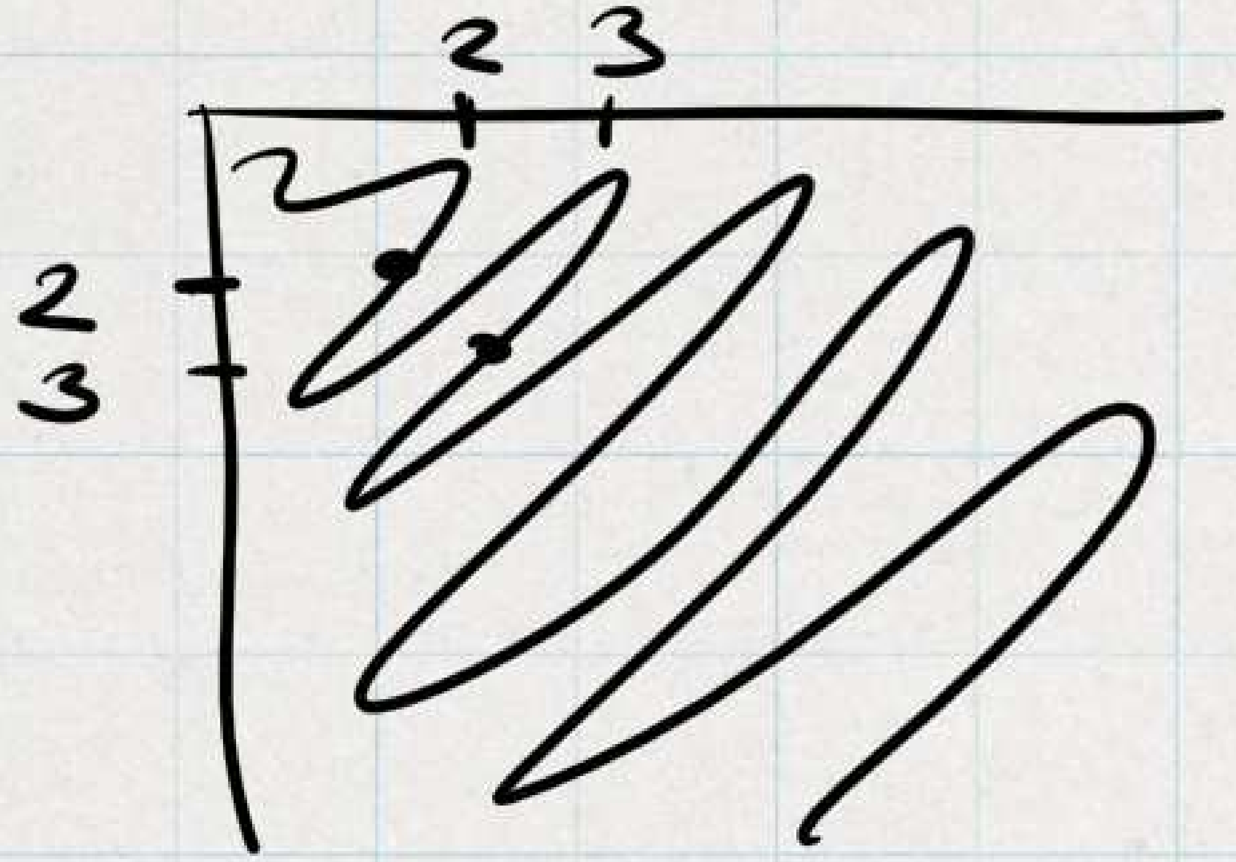
so X is w'oble.

GOAL If X has the property that the $\sim$-eq. class of X contains an ordinal, then we can pick our canonical representative.

What can we say (using Cantor-Schröder-Bernstein, CSB) about the relation $\sim$ in ZF?

You know an explicit bijection

$$\pi : \mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$$



Remember that $\mathbb{Z} = \mathbb{N}\times\mathbb{N}/_{\equiv}$ where $\equiv$ the appr. eq. rel.

<u>injection</u>

$$\begin{aligned} \mathbb{N} &\longrightarrow \mathbb{Z} \\ n &\longmapsto [n,0]_{\equiv} \end{aligned}$$

$$\boxed{|\mathbb{N}| \leq |\mathbb{Z}|}$$

$$\mathbb{Z} \longrightarrow \mathbb{N}$$

use π

$z \in \mathbb{Z} \rightsquigarrow \{(n,m);\ n,m \in \mathbb{N}\ \dots\}$

$$f(z) := \min\{n;\ \pi(n) \in z\}$$

this is an injection since $z, z' \in \mathbb{Z}$

$\Longrightarrow z \cap z' = \emptyset$.

$$\boxed{|\mathbb{Z}| \leq |\mathbb{N}|}$$

CSB

$$\mathbb{N} \sim \mathbb{Z}$$

Do the same with $\mathbb{Q} = \mathbb{Z}\times\mathbb{N}\setminus\{0\} / \equiv$

find $|\mathbb{N}| \leq |\mathbb{Q}|$ and $|\mathbb{Q}| \leq |\mathbb{N}|$

to get $\mathbb{N} \sim \mathbb{Q}$.

What about $\mathbb{R}$?

<u>Theorem (Cantor)</u>. If X is any set, then there is no surjection from X to $P(X)$.

[Clearly, $\underset{x \longmapsto \{x\}}{|X| \leq |P(X)|}$, thus $|X| < |P(X)|$.]

since $X \not\sim P(X)$

<u>Pf.</u> Let $f: X \longrightarrow P(X)$ be any function

To show: there is some set in $P(X)$ not in $\mathrm{ran}(f)$.

$D := \{x \in X;\ x \notin f(x)\}$

DIAGONALISATION

I claim $D \notin \operatorname{ran}(f)$.

Suppose it is: let $d \in X$ s.t. $f(d) = D$.

$d \in D \longleftrightarrow d \notin f(d) = D$. $\lightning$. q.e.d.

[Go back and compare this to the Russell paradox proof.]

What are the reals?

We defined them as subsets of $\mathbb{Q}$. $\mathbb{R} \subseteq P(\mathbb{Q}) \times P(\mathbb{Q})$

If $X \sim Y$, then $P(X) \sim P(Y)$

$f : X \xrightarrow[\text{bij.}]{} Y$

$A \subseteq X$

$\hat{f}(A) := \{f(a);\ a \in A\}$

$\hat{f} : P(X) \xrightarrow[\text{bij.}]{} P(Y)$

So have bij. $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{Q})$.

We get $|\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$.

<u>Lemma</u> If X is any set and $2^X := \{f ; \mathrm{dom}(f) = X \text{ and } \mathrm{ran}(f) \subseteq 2\}$

then $\mathcal{P}(X) \sim 2^X$.

[$A \subseteq X$, let $c_A : X \longrightarrow 2$

$$c_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

CHARACTERISTIC FUNCTION

$A \longmapsto c_A$ is a bijection]

Now define an injection from $2^{\mathbb{N}}$ into $\mathbb{R}$ by $f : \mathbb{N} \longrightarrow 2$

$$x_f := \sum_{i \in \mathbb{N}} \frac{f(i)}{3^i}$$

$\cancel{f(i)3^{-i}}$

INJECTION [why 3 ?]

$$\underline{|P(\mathbb{N})|} = |2^{\mathbb{N}}| \,\underline{\leq |\mathbb{R}|}\, \underline{\leq} |P(\mathbb{Q})| \,\underline{= |P(\mathbb{N})|}$$

CSB : $|P(\mathbb{N})| = |\mathbb{R}|$.

CANONICAL REPRESENTATIVES

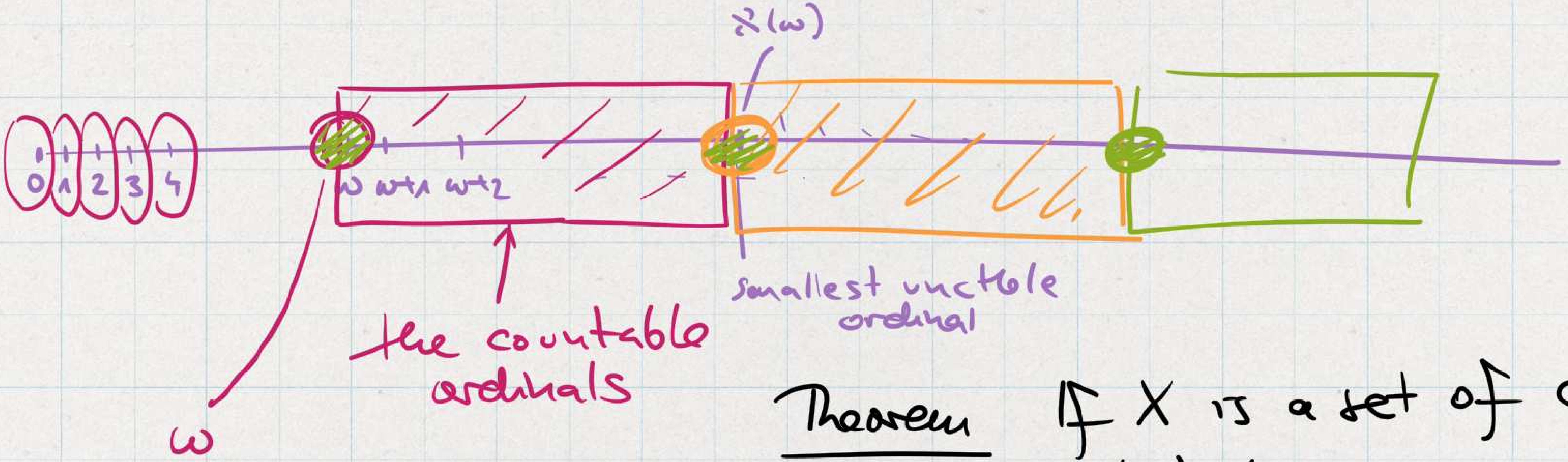
If X is wellorderable, then $|X| := \min\{\alpha ; \alpha \sim X\}$.

Def. An ordinal α is called a cardinal (number) if there is a w'oble set X s.t. $\alpha = |X|$.

Observe If α is a cardinal and $\beta < \alpha$, then there can't be an inj. from α to β.

$$\left.\begin{array}{l} |\alpha| \leq |\beta| \\ |\beta| \leq |\alpha| \end{array}\right\} \xrightarrow{\text{CSB}} |\alpha| = |\beta|,$$

so $\alpha \sim \beta$, contradicting the minimality of α.



<u>CARDINAL</u>
= INITIAL ORDINAL

<u>Theorem</u> If X is a set of cardinals, then $\bigcup X$ is a cardinal.

<u>Pf.</u> $\bigcup X$ is an ordinal

cardinal

Suppose $\beta < \bigcup X \implies \beta \in \bigcup X$
$\implies \exists \kappa \in X \quad \beta \in \kappa$
$\implies |\kappa| \not\leq |\beta|$
$\implies |\bigcup X| \not\leq |\beta|$. q.e.d.

Define

$$\begin{cases} \aleph_0 := \omega \\ \aleph_{\alpha+1} := \aleph(\aleph_\alpha) \\ \aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha \end{cases}$$

ORDINAL OPERATION (normal)

If α is an ordinal, then $\aleph_\alpha$ is a cardinal.
[Induction on α.]

CLAIM If X is w'oble and infinite, then $|X| = \aleph_\alpha$ for some α.

Proof. Let κ be an infinite cardinal. We need to show that there is α s.t. $\kappa = \aleph_\alpha$.

Consider $\{\lambda < \kappa ; \lambda \text{ is an aleph}\}$

$= \underline{\{\aleph_\alpha ; \aleph_\alpha < \kappa\}} \subseteq \kappa$, so a set.

$\{\alpha ; \aleph_\alpha < \kappa\}$ this is a set by Replacement

$\beta := \bigcup \{\alpha ; \aleph_\alpha < \kappa\}$

What is the relationship between $\aleph_\beta$ and κ?

$\aleph_\beta \leq \kappa$ or $\aleph_\beta \geq \kappa$.

$\beta = 0$: $\kappa = \aleph_0 = \omega$.

$\beta = \gamma + 1$: $\aleph_\beta = \underline{\aleph(\aleph_\gamma)}$ with $\underline{\aleph_\gamma < \kappa}$.

$\gamma < \beta$

$\Longrightarrow \aleph_\beta \leq \kappa$, but $\aleph_\beta \not< \kappa$

so $\aleph_\beta = \kappa$.

β limit. So $\{\aleph_\alpha ; \aleph_\alpha < \kappa\}$ has no largest element.

$\aleph_\beta = \bigcup_{\alpha < \beta} \aleph_\alpha$ $\quad \aleph_\beta \leq \kappa$

$\Longrightarrow \aleph_\beta = \kappa$.

q.e.d.

Summary For $\boxed{\text{w'oble sets}}$ X, the right canonical representatives for $|X|$ are the natural numbers and the alephs.

If X is finite $|X| \in \mathbb{N}$.
If X is infinite $\exists \alpha$ ordinal $|X| = \aleph_\alpha$.

Next time AC $\xRightarrow{\text{(Axiom of Choice)}}$ all sets are w'oble!