Set Theory - Fourth Lecture

28 September 2020

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an initial segment of W_2;* proper
(iii) *W_2 is isomorphic to an initial segment of W_1.* proper

FUNDAMENTAL THEOREM ON WELLORDERS

UP TO ISOMORPHISM

~ Linear ordering theorem

$\underline{W_1 \leq^* W_2}$ $:\Longleftrightarrow$ W_1 is iso to an initial segment (not nec. proper) of W_2

$W_1 <^* W_2$ $:\Longleftrightarrow$ W_1 is iso to a proper initial segment of W_2

$\forall W_1, W_2 \; (W_1 \leq^* W_2 \text{ or } W_2 \leq^* W_1)$

$\forall W_1, W_2 \; (W_1 <^* W_2 \text{ or } W_2 <^* W_1 \text{ or } W_1 \cong W_2)$

[MISSING: ANTISYMMETRY]

~ PREORDERS

<u>Theorem</u> If X is a non-empty set of wellorders, then there is $W \in X$ s.t. for all $W' \in X$

$$W \leq^* W'. \quad (*)$$

<u>Pf.</u> $X \neq \emptyset$, let $W \in X$ arbitrary.

If W satisfies $(*)$, done. So assume not.

$$\{W' \in X ;\ W \not\leq^* W'\} \neq \emptyset$$

$$\overset{\text{Fund.}}{\underset{\text{Thm.}}{\Longleftrightarrow}} \{W' \in X ;\ \underline{W' <^* W}\} \neq \emptyset$$

For each such W' find $a(W') \in W$ s.t.

$$W' \cong \overline{I}_{a(W')} \longleftarrow \text{initial segment of } W$$

Consider $\{\underline{x \in W} ;\ \underline{\exists W' \in X}\ I_x \underline{\cong W'}\} \neq \emptyset$.

Let x_0 be the least elt of

$$\{ x \in W ; \exists W' \in X \quad W' \cong I_x \}$$

Find $W' \cong I_{x_0}$. Then this is minimal w.r.t. $\leq^*$ in X. q.e.d.

<u>Corollary</u> If X is any set of wellorders s.t. no two of them are iso, then

$$(X, <^*)$$ is a wellorder.

We would like to have CANONICAL REPRESENTATIVES from the isomorphism classes of wellorders.

ORDER TYPE

ORDINALS

Def. A set α is called an ORDINAL or ORDINAL NUMBER if it is transitive and $(\alpha, \in)$ is a wellorder.

X is transitive if $\forall x, y$ $x \in y \in X \to x \in X$

[eq. $x \in X \Rightarrow x \subseteq X$]

- $\in$ is irreflexive
- $\in$ is transitive
- $\in$ is linear (trichotomous)
- nonempty subsets have least elts

PROPERTIES

(1) $\alpha \notin \alpha$

(2) $x \in \alpha$, then $I_x = \{y \in \alpha ; y \in x\} = \alpha \cap x = x$.

$x \in \alpha \Downarrow x \subseteq \alpha$

[Like in $\mathbb{N}$!!]

(3) If $X \subseteq \alpha$ is a transitive subset, then X is an ordinal.

[Clearly, if $(\alpha, \in)$ is a wellorder, then $(X, \in)$ is a wellorder. $\longrightarrow$ X is an ordinal.]

(4) All elements of ordinals are ordinals.

[From (2), (3): If $x \in \alpha \Longrightarrow x \subseteq \alpha$
$\underset{(2)}{\Longrightarrow} x = I_x \underset{(3)}{\Longrightarrow} x$ is an ordinal.]

(5) Intersections of two ordinals are ordinals.

[$\underbrace{\alpha \cap \beta}_{\text{TRANSITIVE}} \subseteq \alpha$; then apply (3).]

EXAMPLES

1. We proved that each $n \in \mathbb{N}$ is transitive, and $(n, \in)$ is a well-order.
   So each nat. number is an ordinal.
2. $\mathbb{N}$ is an ordinal.
   [Clearly, $(\mathbb{N}, \in)$ is a wellorder.
   But if $x \in n \in \mathbb{N}$, then $x \in \mathbb{N}$.
   [Induction: $Z := \{n \in \mathbb{N};\ n \subseteq \mathbb{N}\}$.]
3. If α is an ordinal, then $S(\alpha) := \alpha \cup \{\alpha\}$ is an ordinal.

$\{\mathbb{N}, S(\mathbb{N}), S(S(\mathbb{N})), \ldots\}$.

This set that we constructed using Repl. consists of ordinals.

Proof of 3. Let α be an ordinal.

$$S(\alpha) = \alpha \cup \{\alpha\}.$$

(c) $\in$ is a wellorder

[Note that $(S(\alpha), \in) \cong (\alpha, \in) \oplus (1, \in)$ by

$$\beta \longmapsto \begin{cases} (\beta, 0) & \text{if } \beta \in \alpha \\ (0, 1) & \text{if } \beta = \alpha \end{cases}$$

We said that sums preserve well-orders. So $(S(\alpha), \in)$ is wellorder.]

(d) $S(\alpha)$ is transitive

[We proved that "x trs $\Longrightarrow S(x)$ trs" in the section covering the natural numbers]

(a) $\in$ is irreflexive

[Since α is an ordinal, $\in$ is irreflexive on elements of α.

Since α is an ordinal, it is irrefl on α by 1.]

(b) $\in$ is transitive

[Let $\beta, \gamma, \delta \in S(\alpha)$.

If $\beta, \gamma, \delta \in \alpha$, then it is just the transitivity the $\in$-relation on α.

If one of them is $= \alpha$, say $\beta \in \gamma \in \delta = \alpha$, then it is the transitivity of the set α.]

Lemma 2.11.

(i) $0 = \emptyset$ is an ordinal. ✓

(ii) *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.* ✓

(iii) *If $\alpha \neq \beta$ are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.*

(iv) *If α, β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$.*

$\subsetneq$ $\subseteq$ $\subseteq$

(iii) If α, β ordinals $\alpha \subsetneq \beta$, then $\alpha \in \beta$.

<u>Proof</u> Consider $X := \beta \setminus \alpha \neq \emptyset$.

Let γ be least s.t. $\gamma \in X$. [β is an ordinal.]

<u>Claim</u> $\alpha = \gamma$. [This proves the claim.]

Suppose $\delta \in \alpha$ and $\delta \notin \gamma$.

$\delta, \gamma \in \beta \xrightarrow{\subseteq \beta} \delta \in \gamma$ or $\underline{\delta = \gamma}$ or $\underline{\gamma \in \delta}$

<u>Case 1</u> $\delta = \gamma \implies \gamma \in \alpha$ ↯

<u>Case 2</u> $\gamma \in \delta \in \alpha \underset{\text{trs of } \alpha}{\implies} \gamma \in \alpha$ ↯

Suppose $\delta \in \gamma$ and $\delta \notin \alpha$.
Contradiction to minimality of γ.

q.e.d.

Lemma 2.11.

(i) $0 = \emptyset$ *is an ordinal.*
(ii) *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*
(iii) *If $\alpha \neq \beta$ are ordinals and $\alpha \subset \beta$, then $\alpha \in \beta$.*
(iv) *If α, β are ordinals, then either $\alpha \subset \beta$ or $\beta \subset \alpha$.*

$\subseteq$ behaves like a linear order on ordinals

$\in$ behaves like a strict linear order on ordinals.

(iv).

$\alpha \cap \beta =: \gamma$ is an ordinal, as proved earlier.

$\gamma \subseteq \alpha$ and $\gamma \subseteq \beta$.

Case 1 $\gamma = \alpha \implies \alpha \cap \beta = \alpha \implies \alpha \subseteq \beta$.

Case 2 $\gamma = \beta \implies \alpha \cap \beta = \beta \implies \beta \subseteq \alpha$.

Case 3
$$\left.\begin{array}{l} \gamma \subsetneqq \alpha \overset{(iii)}{\implies} \gamma \in \alpha \\ \gamma \subsetneqq \beta \overset{(iii)}{\implies} \gamma \in \beta \end{array}\right\} \implies \gamma \in \underbrace{\alpha \cap \beta}_{= \gamma}.$$

Contradicts property 1.

q.e.d.

Corollary 1 Ordinals are unique in their ordertype:

$$(\alpha, \in) \cong (\beta, \in) \implies \alpha = \beta.$$

Suppose α, β are isomorphic. W.l.o.g., $\alpha \subseteq \beta$ (iv) 2.11. If $\alpha \neq \beta$, then $\alpha \in \beta$ (iii) 2.11.

$\implies \alpha = I_\alpha$. So β cannot be isomorphic to α as wellorders are never iso to proper initial segments.

Corollary 2 Transitive sets of ordinals are ordinals.

[X be a transitive set of ordinals: to show: $(X, \in)$ is a wellorder.

This follows from Lemma 2.11 (linearity) + nonempty sets of wellorders have least elements.]

Corollary 3 If A is a set of ordinals, then $\bigcup A$ is an ordinal.

Example for Cor. 3 2 or

$A := \{\mathbb{N}, S(\mathbb{N}), S(S(\mathbb{N})), \dots\}$

$x_0 := \mathbb{N}$
$x_{n+1} := S(x_n)$
$\{x_n ; n \in \mathbb{N}\}$

$\bigcup A = \mathbb{N} \cup A$

By cor. 3, this is an ordinal.

Corollary 4 There is no set of all ordinals.

This means that there are "lots" of ordinals.

[Suppose Ω is the set of all ordinals.
Since elements of ordinals are ordinals, Ω is transitive.
By Cor. 2, Ω is an ordinal.
$\Longrightarrow \Omega \in \Omega$. Contradicts (1).]

CONCRETE ORDINALS.

$0, 1, 2, 3, 4, 5, \ldots$

$\mathbb{N}$ If we refer to the natural numbers as an ordinal, we usually call it ω [omega].

$$\omega+1 = S(\mathbb{N}) = \omega \cup \{\omega\} \equiv \omega+1$$

$$= \mathbb{N} \cup \{\mathbb{N}\}$$

If α is an ordinal, write $\alpha+1$ for $S(\alpha)$.

$$(\omega+1)+1 = S(S(\mathbb{N}))$$

$$\boxed{\begin{array}{l} x_0 := \omega \\ x_{n+1} := x_n + 1 \end{array}}$$

Function x with $\mathrm{dom}(x) = \mathbb{N}$

$\mathrm{ran}(x) = A$ [from last page]

$\bigcup \mathrm{ran}(x)$ is an ordinal

$$\cong \mathbb{N}$$

$$0\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ \ldots\ \ \ \omega\ \ \omega+1\ \ \omega+2\ \ \omega+3\ \ \ldots\ \ \omega+\omega\ \ S(\omega+\omega)$$

$$(\omega+\omega)+1$$

$$\mathbb{N}$$

$\mathrm{ran}(x)$

$\bigcup \mathrm{ran}(x)$

$$\cong (\mathbb{N},<) \oplus (\mathbb{N},<)$$

The orange ones are successors of something:

e.g. $4 = S(3)$

$\omega+3 = S(\omega+2)$

The yellow ones are not

A good name for $\bigcup \mathrm{ran}(x)$ might be

$\omega+\omega$.

Def. An ordinal α is called a SUCCESSOR ORDINAL if there is $\beta \in \alpha$ s.t.

$\alpha = S(\beta) . = \beta + 1$.

Otherwise, we call α a LIMIT ORDINAL.

[Sometimes 0 is not considered a limit.]

We write $\alpha < \beta$ $:\Longleftrightarrow$ $\alpha \in \beta$.

Proposition If α is an ordinal, then $^{(1)}$ there is $\beta_{(2)} > \alpha$ s.t. β is a successor ordinal and there is $\gamma > \alpha$ s.t. γ is a limit ordinal.

[$\beta := \alpha + 1$; $x_0 := \alpha$, $x_{n+1} := (x_n) + 1$; $\gamma := \bigcup \mathrm{ran}(x)$.

Claim: γ is a limit. Suppose $\gamma = \beta + 1$ with $\beta \in \gamma$.

$\beta \in x_n$ for $n \in \mathbb{N}$

$\beta + 1 \in x_{n+1} \subseteq \gamma \to \gamma \in \gamma$ ↯ q.e.d.]

Q. Are there any uncountable wellorders?

[$W \oplus W'$ both preserve
$W \otimes W'$ countability]

Theorem (Hartogs)

If X is any set, then there is an ordinal α s.t. α does not inject into X.

Application $X := \mathbb{N}$. Then the theorem gives me an uncountable ordinal α.

The smallest such α constructed in the proof is also called the HARTOGS ALEPH of X. $\aleph$

$$h(X) = \aleph(X)$$

ALEPH

Proof of Hartogs's Thm.

Fix X and consider $\mathcal{R}_X := \{(A,R) ; A \subseteq X \ \& \ R \subseteq A \times A\}$

In Zermelo set theory, $\mathcal{R}_X$ is a set.

$$(A,R) \longmapsto \begin{cases} 0 & \text{if } (A,R) \text{ is not a wellorder} \\ \alpha & \text{if } (A,R) \cong (\alpha,\in) \end{cases}$$

[We need that every wellorder is isomorphic to an ordinal. Proof next.]

This is functional.

By Axiom of Repl., there is a set

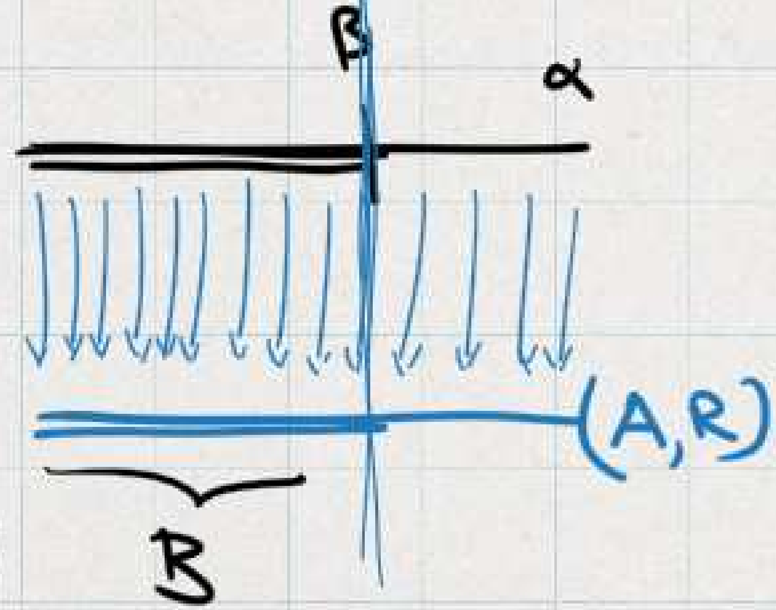
$$H := \{\alpha ; \exists (A,R) \in \mathcal{R}_X \ (\alpha,\in) \cong (A,R)\}$$

This is the collection of ordinals that can be encoded as a subset of X.

__Claim__ H is transitive.

Suppose $\alpha \in H$: $(\alpha, \in) \cong (A, R)$ with $A \subseteq X$, $R \subseteq A \times A$

$\beta < \alpha$



$f \upharpoonright \beta$ is a iso between $(\beta, \in)$ and (B, R').

$B := f[\beta]$

$B \subseteq A \subseteq X$

$R' := R \cap B \times B$

$\Longrightarrow \beta \in H$.

So, H is a transitive set of ordinal, thus an ordinal.

$$H := h(X) := \aleph(X).$$

Claim 2 H does not inject into X.

Suppose it does: $g : H \longrightarrow X$ injective.

$(H, \in)$

$A := \operatorname{ran}(g)$. Then $g : H \longrightarrow A$ is bijective.

Define $g(\alpha) R g(\beta) :\Longleftrightarrow \alpha \in \beta$

Then $(A, R) \cong_g (H, \in)$. But by def. of H, $H \in H$. ↯ q.e.d.

[Modulo ex. of ordinal iso to wellorders.]

Theorem If W is a wellorder, then there is a unique
ZF ordinal α s.t. $W \cong (\alpha, \in)$.

[REPRESENTATION THEOREM FOR WELLORDERS]

Proof. Define on W by recursion the following function: MOSTOWSKI COLLAPSE

$$F(x) := \{F(y) ; y < x\}.$$
$$= \operatorname{ran}(F \upharpoonright I_x)$$

Consider $\operatorname{ran}(F) = A$. If $y < x \iff F(y) \in F(x)$. By construction!

Note that $F : (W, <) \longrightarrow (A, \in)$ is an order pres. bijection.

Thus $(A, \in)$ is a wellorder. $\Big\} \Longrightarrow A$ is an ordinal.

By construction A is transitive: $z \in F(x) \in A$
$\rightarrow z = F(y)$ some $y < x$.

q.e.d.

STRUCTURE OF THE ORDINALS

The ordinals are not a set: it makes no sense to say "the ordinal are a well order".

But: they behave like a wellorder:

α, β ordinals: either $\alpha \in \beta$ or $\alpha = \beta$ or $\beta \in \alpha$

$$\underbrace{\alpha \in \beta}_{\alpha < \beta} \qquad \underbrace{\beta \in \alpha}_{\beta < \alpha}$$

If X is a non-empty set of ordinals, then X has a least element.

<u>Thus</u>: we can do induction & recursion on ordinals.

TRANSFINITE INDUCTION

If Φ is any formula and

1. $\Phi(0)$ holds
2. if $\Phi(\alpha)$ holds, then $\Phi(\alpha+1)$ holds
3. if λ is a non-zero limit and f.a. $\alpha<\lambda$ $\Phi(\alpha)$ holds, then $\Phi(\lambda)$

Then $\Phi(\alpha)$ for all ordinals α.

<u>Proof.</u> Suppose not, so there is some α s.t. $\neg\Phi(\alpha)$.
There must be a least ordinal α with this property:
<u>Case 1</u>. α is already least : DONE.

Case 2 α is not least

$$B := \{\beta \in \alpha ; \neg\Phi(\beta)\} \neq \emptyset$$

So pick β_0 least in B. Then is the least ord. with $\neg\Phi(\beta_0)$.

Now pick that least ctrex and check that 1, 2 & 3 make this impossible! q.e.d.

TRANSFINITE RECURSION

Fix x_0 arbitrary.

Let Φ be a functional formula. Write $F(x)$ for unique y s.t. $\Phi(x,y)$.

Then there is an operation G on the ordinals

$$G(0) := x_0$$

$$G(\alpha+1) := F(G(\alpha+1))$$

$$G(\lambda) := F(G \upharpoonright \lambda)$$

[if λ is non-zero limit]

MAIN APPLICATION ORDINAL ARITHMETIC

Fix α

	ADDITION	MULTIPLICATION	EXPONENTIATION
	$\alpha+0 := \alpha$	$\alpha\cdot 0 := 0$	$\alpha^0 := 1$
	$\alpha+(\beta+1) := (\alpha+\beta)+1$	$\alpha\cdot(\beta+1) := \alpha\cdot\beta + \alpha$ (uses addition)	$\alpha^{\beta+1} := \alpha^\beta\cdot\alpha$
λ is non-zero limit	$\alpha+\lambda := \bigcup_{\beta<\lambda} \alpha+\beta$	$\alpha\cdot\lambda := \bigcup_{\beta<\lambda}\alpha\cdot\beta$	$\alpha^\lambda := \bigcup_{\beta<\lambda}\alpha^\beta$

GRASSMANN RECURSION EQUATIONS

NOTE: ASYMMETRIC: Fixed α, recursion in 2nd coordinate. It needs to be checked that $+, \cdot$ on $\mathbb{N}$ are commutative.

Even more asymmetric.

In the recursion $\{\alpha+\beta ; \beta<\lambda\}$ is a set by Repl. and thus this is an ordinal.

ASYMMETRY:

The asymmetry in the GRASSMANN case for $\mathbb{N}$ lies in which of the variables we do recursion over (the right one).

The asymmetry in the ordinal case lies in addition in the determination whether we are in the successor or limit case. (the right ordinal).

CALCULATIONS

+ ordinal addition
+ successor op.

$$\underline{\omega + 1} = \omega + (0+1) = S(\omega + 0) = S(\omega) = \underline{\omega + 1} > \omega.$$

COMMUTATIVITY FAILS!

$$(\omega, \in) \oplus (1, \in) \cong (\omega+1, \in).$$

$$\underbrace{\bullet \bullet \bullet \bullet \cdots\cdots}_{\mathbb{N}} \quad \bullet \qquad \not\cong$$

$$(1, \in) \oplus (\omega, \in) \cong (\omega, \in)$$

$$\bullet \quad \underbrace{\bullet \bullet \bullet \bullet \cdots\cdots}_{\mathbb{N}}$$

$1 + \omega$ LIMIT

$$1 + \omega = \bigcup_{\beta < \omega} 1 + \beta = \bigcup \{x ; \exists \beta < \omega \ x = 1 + \beta\} = \bigcup \{n+1 ; n \in \mathbb{N}\} = \mathbb{N} = \omega.$$

[LEMMA
$\alpha \cdot 1 = \alpha$]

$$\alpha \cdot 1 = \alpha(0+1) = \alpha \cdot 0 + \alpha = 0 + \alpha$$

[LEMMA
$0 + \alpha = \alpha$]

$$\boxed{\begin{array}{l}(\omega,<) \otimes (2,<) \\ (2,<) \otimes (\omega,<)\end{array}}$$

$$\omega \cdot 2 = \omega \cdot (1+1) = \omega \cdot 1 + \omega = \omega + \omega = \bigcup_{n\in\omega} \omega + n$$

$\sim$ $(\omega, \in) \oplus (\omega, \in)$

$$2 \cdot \underline{\underline{\omega}} = \bigcup_{n\in\omega} 2 \cdot n = \bigcup \{x ;\ \exists n (n\in\omega \wedge x = 2n)\} = \omega$$

Some Alephs :