Set Theory III

21 September 2020

Axioms of Zermelo-Fraenkel

1.1. *Axiom of Extensionality.* *If X and Y have the same elements, then $X = Y$.*

1.2. *Axiom of Pairing.* *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. *Axiom Schema of Separation.* *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. *Axiom of Union.* *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. *Axiom of Power Set.* *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. *Axiom of Infinity.* *There exists an infinite set.*

1.7. *Axiom Schema of Replacement.* *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. *Axiom of Regularity.* *Every nonempty set has an $\in$-minimal element.*

1.9. *Axiom of Choice.* *Every family of nonempty sets has a choice function.*

The theory with axioms 1.1–1.8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of Choice.

FST

Z

ZERMELO

ZF ZERMELO-FRAENKEL

FST :

"abstract mathematics"

Z :

$\mathbb{N}$

GT#3:

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Z is enough for most of mathematics

Adolf Fraenkel (1922), Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre, *Mathematische Annalen* 86, p. 230–237

Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre.

Von

Adolf Fraenkel in Marburg.

Wenn man die Cantorsche Mengenlehre[1]), unter Ausscheidung der Antinomien und unter Verzicht auf die ihnen Raum gebende Cantorsche Mengendefinition, auf mathematisch befriedigende Grundlagen stellen will, so kommt vorläufig nur die von Herrn Zermelo gegebene Begründung[2]) in Frage. Einige das Grundgerüst dieser Begründung betreffende und z. T es modifizierende Bemerkungen bilden den Inhalt der folgenden Zeilen; eine ausführlichere und zusammenhängende Erörterung des hier in einigen Kernpunkten berührten Fragenkomplexes bleibt einem weiteren Aufsatz vorbehalten, in dem eine endgültige axiomatische Begründung versucht wird. Die überaus scharfsinnigen Untersuchungen Zermelos sollen hierdurch nicht umgestoßen, sondern nur vervollständigt und befestigt werden, u. a. auch nach der Richtung der bisher nicht gelungenen Klärung der *Unabhängigkeit der Axiome*.

I. *Die sieben Zermeloschen Axiome reichen nicht aus zur Begründung der Mengenlehre.*

Zum Nachweis dieser Behauptung diene etwa das folgende einfache Beispiel: Es sei Z_0 die Z., S. 267, definierte und als existierend nachgewiesene Menge (Zahlenreihe); die Potenzmenge $\mathfrak{U}Z_0$ (Menge aller Untermengen von Z_0) werde mit Z_1, $\mathfrak{U}Z_1$ mit Z_2 bezeichnet usw. Dann gestatten die Axiome, wie deren Durchmusterung leicht zeigt, nicht die

[1]) Vom Standpunkt Kronecker-Brouwer-Weyl — für die Mengenlehre kommt wesentlich Brouwer in Betracht — wird hier abgesehen; die Differenzen zwischen dieser Auffassung und der heute in der Analysis üblichen, die an die Namen Weierstraß und Cantor geknüpft werden kann, dürften mindestens noch geraume Zeit weiterbestehen.

[2]) Math. Ann. 65 (1908), S. 261–281. Zitiert als Z.

EXAMPLE.

In Z, we have $\mathbb{N}$;

we have operation

$$x \longmapsto x \cup \{x\} = S(x)$$

$$x \longmapsto \mathcal{P}(x)$$

In fact $\mathbb{N} \neq S(\mathbb{N})$

$$S(\mathbb{N}) \neq S(S(\mathbb{N}))$$

etc.

$$\begin{cases} N_0 := \mathbb{N} \\ N_{i+1} := S(N_i) = N_i \cup \{N_i\} \end{cases}$$

by RECURSION

G, $\mathrm{dom}(G) = \mathbb{N}$

Consider $\mathrm{ran}(G)$

$\{\mathbb{N}, S(\mathbb{N}), S(S(\mathbb{N})), S(S(S(\mathbb{N}))), \dots\}$

PROOF OF RECURSION

1. Define the right notion of germ.
2. Prove: any two germs coincide on the intersection of their domains
3. Prove: for any $n \in \mathbb{N}$, there is a germ g with $n \in \operatorname{dom}(g)$.
4. Use Separation on $\mathbb{N} \times Z$ for the fmla

$$\Phi(x,y) :\Longleftrightarrow \exists g \text{ germ } g(x)=y.$$

Need some component that gives us Z in order to apply Separation.

Replacement Schema

If a class F is a function, then for every set X, $F(X)$ is a set.

For each formula $\varphi(x, y, p)$, the formula (1.7) is an Axiom (of Replacement):

$$(1.7) \quad \underline{\forall x\, \forall y\, \forall z\, (\varphi(x, y, p) \wedge \varphi(x, z, p) \to y = z)} \\ \to \forall X\, \exists Y\, \forall y\, (y \in Y \leftrightarrow (\exists x \in X)\, \varphi(x, y, p)).$$

As in the case of Separation Axioms, we can prove the version of Replacement Axioms with several parameters: Replace p by $\underbrace{p_1, \ldots, p_n}_{n \text{ parameters}}$.

$$\forall x \forall y \forall z \ \varphi(x, y, p) \wedge \varphi(x, z, p) \longrightarrow y = z$$ FUNCTIONALITY

"the formula φ behaves like a $\underbrace{\text{function}}_{\text{partial}}$".

STRENGTHENING

$$\forall x \forall y \forall z \ \varphi(x, y, p) \wedge \varphi(x, z, p) \longrightarrow y = z$$
$$\wedge \ \forall x \exists y \ \varphi(x, y, p)$$

TOTAL FUNCTIONALITY

Replacement Schema

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For each formula $\varphi(x, y, p)$, the formula (1.7) is an Axiom (of Replacement):

$$(1.7)\quad \forall x\,\forall y\,\forall z\,(\varphi(x,y,p) \wedge \varphi(x,z,p) \to y = z) \to \forall X\,\exists Y\,\forall y\,(y \in Y \leftrightarrow (\exists x \in X)\,\varphi(x,y,p)).$$

As in the case of Separation Axioms, we can prove the version of Replacement Axioms with several parameters: Replace p by $p_1, \ldots, p_n$.

"Everything that behaves like a function is a function".

WEAKENING:

$$\Big[\forall x\,\forall y\,\forall z\;\varphi(x,y,p) \wedge \varphi(x,z,p) \to y = z \;\wedge\; \forall x\,\exists y\;\varphi(x,y,p)\Big] \longrightarrow \forall X\,\exists Y\,\forall y\,\big(y \in Y \longleftrightarrow \exists x \in X(\varphi(x,y,p))\big)$$

OBSERVATION In Z, the weaker version of Repl. implies the stronger.

Suppose φ is functional, but not nec. total. If X is given, need some Y s.t. $\forall y\,(y \in Y \longleftrightarrow \exists x \in X(\varphi(x,y,p)))$.

If $\underline{Y} \neq \emptyset$, then let $y_0 \in \underline{Y}$ and define

formula $\varphi^*(x,y,p) :\longleftrightarrow \varphi(x,y,p) \vee (\forall z\, \neg\varphi(x,z,p) \wedge y = y_0)$

TOTAL FUNCTIONAL

Then apply weak Repl. to φ^* and get ...

$$\underline{Y} \cup \{y_0\} = \underline{Y}$$

If the desired set $\underline{Y}$ is empty, then Z proves the existence of $\underline{Y}$ anyway.

Why is it called REPLACEMENT?

$$X = \{x \in X \,;\, x \in X\}$$

If F is a function with $X \subseteq \mathrm{dom}(F)$.

$$\{F(x) \,;\, x \in X\}$$

RECURSION THEOREM (without fixed range).

Suppose $\varphi(x,y,p)$ is a total functional formula & x_0 arbitrary.

Let's write $F(x)$ for the unique y s.t. $\varphi(x,y,p)$.

By this notation, we do NOT mean to imply that F is a set.

Then there is a unique G with $\mathrm{dom}(G) = \mathbb{N}$

and
$$G(0) := x_0$$
$$G(n+1) := F(G(n)).$$

Proof.

(1) Define germ (as before)

(2) Prove that germs agree on their common domain (as before)

(3) Prove that every $n \in \mathbb{N}$ is in the domain of a germ (as before)

(4) Consider

$$\Phi(n, x, p) :\Longleftrightarrow \exists g \text{ germ } n \in \mathrm{dom}(g) \wedge g(n) = x$$

Φ is total functional formula, so apply Repl. to Φ and get a set Y s.t. Y contains all ranges of all germs

$\longrightarrow$ Separate G from $\mathbb{N} \times Y$. q.e.d.

Now go back to Fraenkel's example:

$$\varphi(x,y) := y = s(x)$$

Clearly total functional.

$$y = x \cup \{x\}.$$

uniqueness uses the axioms of FST

Recursion Theorem $\Longrightarrow$

$\exists G$ with $\mathrm{dom}(G) = \mathbb{N}$

and $G(0) = \mathbb{N}$

$G(n+1) = S(G(n))$.

$\mathrm{ran}(G) = \{\mathbb{N}, S(\mathbb{N}), S(S(\mathbb{N})), \ldots\}$.

This is a function!

$\mathrm{dom}(G)$, $\mathrm{ran}(G)$ are sets

The Axiom of Regularity states that the relation $\in$ on any family of sets is well-founded:

Axiom of Regularity. *Every nonempty set has an $\in$-minimal element:*

$$\forall S \, (S \neq \emptyset \to (\exists x \in S) \, S \cap x = \emptyset).$$

FOUNDATION

$x = \{x\}$

$x \in x$

In $\mathbb{N}$, that's not the case: $n \notin n$. (if $n \in \mathbb{N}$).

The axioms $ZF_0 := Z + \text{Repl.}$ do not answer the question

? Is there a set x s.t. $x \in x$?

We want set theory to answer this.

REGULARITY / FOUNDATION gives an answer.

$ZF := ZF_0 + \text{Regularity}.$

<u>Theorem</u> (ZF). There is no x s.t. $x \in x$.

<u>Proof.</u> Let $X := \{x\}$.

Apply Reg. to $X \neq \emptyset$. Find $z \in X$ s.t.

$$\left.\begin{array}{l} z \cap X = \emptyset. \\ \text{Clearly } z = x. \end{array}\right\} \Longrightarrow \underset{x}{\underset{\cup\!\!\!\shortmid}{x}} \cap \underset{x}{\underset{\cup\!\!\!\shortmid}{X}} = \emptyset$$

$\lightning$ q.e.d.

The Axiom of Regularity states that the relation $\in$ on any family of sets is well-founded:

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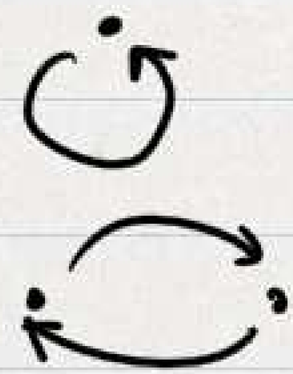
Why not just $\forall x\,(x \notin x)$?

Because then, e.g., $\exists x \exists y\,(x \in y \wedge y \in x)$, remains undecided.

REG solves all types of "circularity" in the negative.



__Later__ : More on the structural consequences of the Axiom of Regularity.

Definition 2.1. A binary relation $<$ on a set P is a *partial ordering* of P if:

(i) $p \not< p$ for any $p \in P$; IRREFLEXIVE
(ii) if $p < q$ and $q < r$, then $p < r$. TRANSITIVE

$(P, <)$ is called a *partially ordered set*. A partial ordering $<$ of P is a *linear ordering* if moreover

TRICHOTOMY

(iii) $\boxed{p < q \text{ or } p = q \text{ or } q < p}$ for all $p, q \in P$.

If $<$ is a partial (linear) ordering, then the relation $\leq$ (where $p \leq q$ if either $p < q$ or $p = q$) is also called a partial (linear) ordering (and $<$ is sometimes called a *strict* ordering).

Definition 2.2. If $(P, <)$ is a partially ordered set, X is a nonempty subset of P, and $a \in P$, then:

a is a *maximal* element of X if $a \in X$ and $(\forall x \in X)\, a \not< x$;
a is a *minimal* element of X if $a \in X$ and $(\forall x \in X)\, x \not< a$;
a is the *greatest* element of X if $a \in X$ and $(\forall x \in X)\, x \leq a$;
a is the *least* element of X if $a \in X$ and $(\forall x \in X)\, a \leq x$;
a is an *upper bound* of X if $(\forall x \in X)\, x \leq a$;
a is a *lower bound* of X if $(\forall x \in X)\, a \leq x$;
a is the *supremum* of X if a is the least upper bound of X;
a is the *infimum* of X if a is the greatest lower bound of X.

The supremum (infimum) of X (if it exists) is denoted $\sup X$ ($\inf X$). Note that if X is linearly ordered by $<$, then a maximal element of X is its greatest element (similarly for a minimal element).

If $(P, <)$ and $(Q, <)$ are partially ordered sets and $f : P \to Q$, then f is *order-preserving* if $x < y$ implies $f(x) < f(y)$. If P and Q are linearly ordered, then an order-preserving function is also called *increasing*.

A one-to-one function of P onto Q is an *isomorphism* of P and Q if both f and f^{-1} are order-preserving; $(P, <)$ is then *isomorphic* to $(Q, <)$. An isomorphism of P onto itself is an *automorphism* of $(P, <)$.

1st protagonist:

ORDINAL NUMBERS

Jech's Chapter 2.

strict	non-strict
$<$	$\leq$
IRREFLEXIVE	REFLEXIVE
TRANSITIVE	TRANSITIVE
	ANTI-SYMMETRIC

$< \;\rightsquigarrow\; x \leq y :\iff x < y$ or $x = y$

$x < y :\iff x \leq y$ and $x \neq y \;\longleftarrow\; \leq$

$x \subset y$ $\quad x \subsetneq y$

$x \subseteq y$

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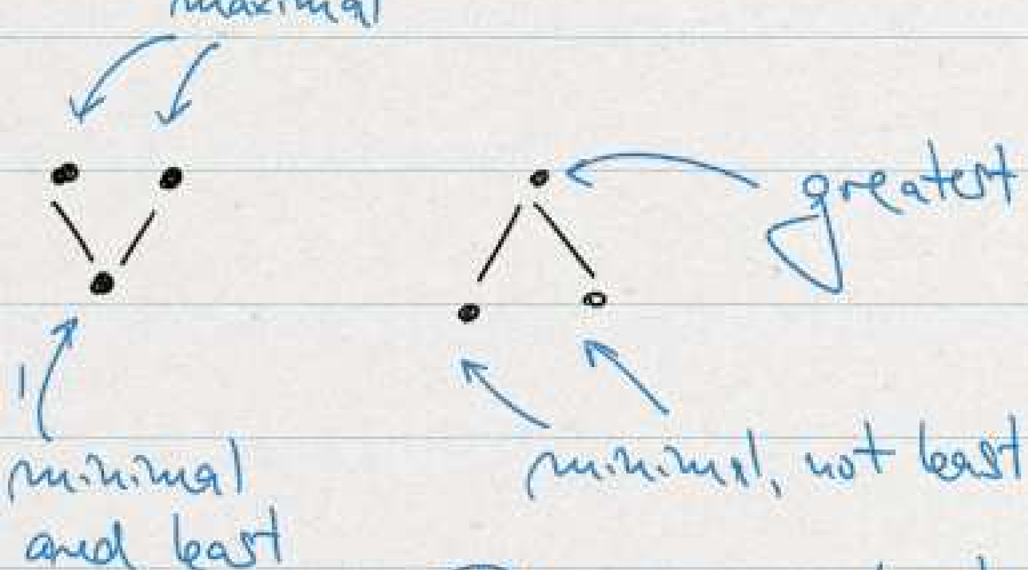


There is no greatest elt.

There is no least element.

If $(X,<)$ is a total order, then greatest & maximal and least & minimal coincide!

For the first section, we saw $(\mathbb{N},<)$ where $n<m$ was defined as $n \in m$.

In HW#2, you proved that this is a total order.
linear

The least number principle for $\mathbb{N}$ says:

<u>Thm</u> Every nonempty subset of $\mathbb{N}$ has a least element.

<u>Proof.</u> Reformulate to:

If $\underline{X \subseteq \mathbb{N} \text{ has no least element}}$, then $X = \emptyset$.

$[\mathbb{N} \setminus X = \mathbb{N}]$.

So let's assume and show that $Z := \mathbb{N} \setminus X$ is inductive.

Instead, look at $Z' := \{x \in \mathbb{N} ; \forall z \leq x \, (z \in Z)\}$

Clearly $Z' \subseteq Z$. So if Z' is inductive, then $\mathbb{N} = Z' \subseteq Z \subseteq \mathbb{N}$.

(1) $0 \in Z'$: If $0 \in X$ then 0 is the least elt. of X, contra!

So $0 \notin X \Rightarrow 0 \in Z \Rightarrow 0 \in Z'$.

(2) Suppose $x \in Z'$. (by def. $\forall z \leq x \, (z \in Z)$) $\longleftrightarrow \forall z \leq x \, (z \notin X)$

If $s(x) \notin Z'$, then $s(x) \notin Z$. Thus $s(x)$ is the least elt. X, contra!

$\longleftrightarrow s(x) \in X$

q.e.d.

DEF 2.3 A linear ordering $(P,<)$ is called a WELL-ORDERING if every nonempty subset of P has a $<$-least element.

EXAMPLES & NON-EXAMPLES

$(\mathbb{N},<)$ is a wellordering

$(n,<)$ is a wellordering

$\mathbb{Z}$ are not a wellordering [they have no least-element]

$\mathbb{Q}$, $\mathbb{R}$

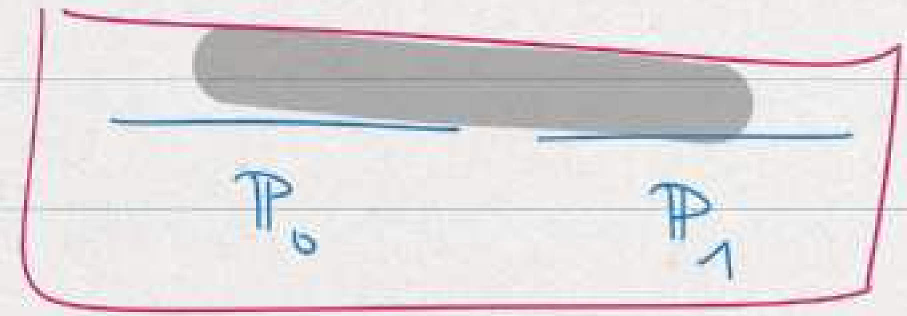
$$P := \{0\} \cup \underbrace{\left\{\frac{1}{2^n};\, n\in\mathbb{N}\right\}}$$

has a least element

$\left\{\frac{1}{2^n};\, n\in\mathbb{N}\right\} \subseteq P$ with no least element.

More examples

$\mathbb{P}_0 := (P_0, <_0)$ $\quad$ $\mathbb{P}_1 := (P_1, <_1)$ two linear orders



Define the order sum $\mathbb{P}_0 \oplus \mathbb{P}_1$.

DISJOINT UNION of two sets P_0, P_1:

$$\{0\} \times P_0 \cup \{1\} \times P_1$$

$P := \{0\} \times P_0 \cup \{1\} \times P_1$ and $(i, p) < (j, q) :\Longleftrightarrow$
$(i = 0$ and $j = 1)$ or
$(i = j = 0$ and $p <_0 q)$ or
$(i = j = 1$ and $p <_1 q)$.

If $\mathbb{P}_0, \mathbb{P}_1$ are linear then so is $\mathbb{P}_0 \oplus \mathbb{P}_1$.

<u>CLAIM</u> If $\mathbb{P}_0, \mathbb{P}_1$ are wellorders, then so is $\mathbb{P}_0 \oplus \mathbb{P}_1$.

Proof. If $Z \subseteq \{0\} \times P_0 \cup \{1\} \times P_1$ is non-empty, then there are two cases:

Case 1 $Z \cap \{0\} \times P_0 \neq \emptyset$.

In that case $\underset{\neq \emptyset}{\bar{Z}} := \{p \,;\, (0,p) \in Z\} \subseteq P_0$

Let $\bar{p}$ be least in $\bar{Z}$, then $(0,\bar{p})$ is least in Z.

Case 2 $Z \cap \{0\} \times P_0 = \emptyset$

$\Longrightarrow Z \cap \{1\} \times P_1 \neq \emptyset$ because of that

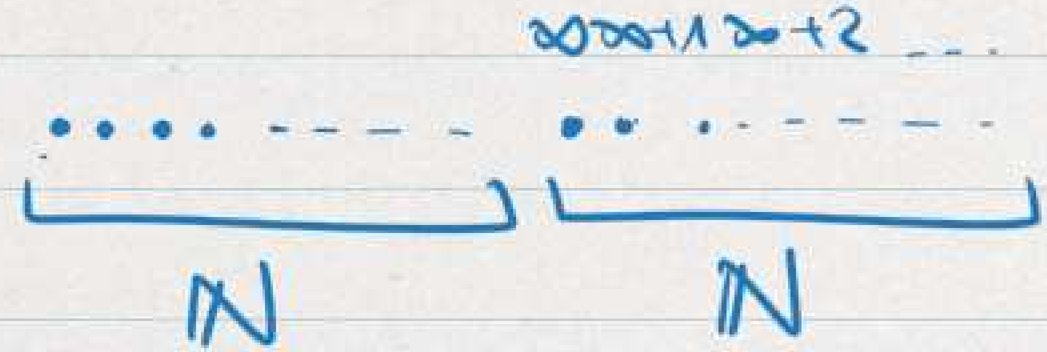
$\bar{Z} := \{p \,;\, (1,p) \in Z\} \subseteq P_1$

$\neq \emptyset$ Let $\bar{p}$ be least in $\bar{Z}$, then $(1,\bar{p})$ is least in Z.

q.e.d.

Apply this to $(\mathbb{N}, <)$:

$$(\mathbb{N}, <) \oplus (\mathbb{N}, <)$$



$$\big((\mathbb{N}, <) \oplus (\mathbb{N}, <)\big) \oplus (\mathbb{N}, <)$$

or n copies of the natural numbers.

If $\mathbb{P}_0, \mathbb{P}_1$ are linear orders, we define the <u>product</u> $\mathbb{P}_0 \otimes \mathbb{P}_1$ as follows:

$P := P_0 \times P_1$

$$(p, p') < (q, q') :\Longleftrightarrow \text{if } p < q \text{ or } (p = q \text{ and } p' < q').$$

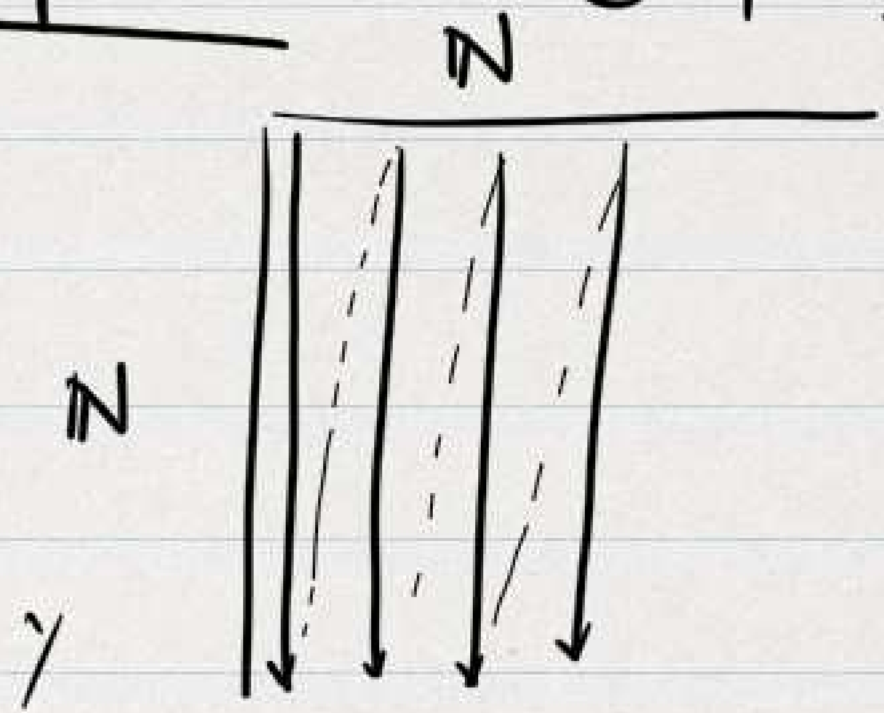
In general, $\mathbb{P}_0 \otimes \mathbb{P}_1$ is a linear order.

Claim If $\mathbb{P}_0, \mathbb{P}_1$ are wellorders, then
$\mathbb{P}_0 \otimes \mathbb{P}_1$ is a wellorder.

Proof. First find least $\bar{p}$ occurring in the first component, then find least $\bar{q}$ s.t. $(\bar{p}, \bar{q}) \in Z$.

Examples $(\mathbb{N}, <) \otimes (\mathbb{N}, <)$

$\mathbb{N} \times \mathbb{N}$



$$\begin{array}{cc} (0,0) & (1,0) \\ (0,1) & (1,1) \\ (0,2) & (1,2) \\ (0,3) & \vdots \\ \vdots & \end{array}$$

Basic properties of wellorders

Lemma 2.4. *If $(W,<)$ is a well-ordered set and $f: W \to W$ is an increasing function, then $f(x) \geq x$ for each $x \in W$.*

Corollary 2.5. *The only automorphism of a well-ordered set is the identity.*

Corollary 2.6. *If two well-ordered sets W_1, W_2 are isomorphic, then the isomorphism of W_1 onto W_2 is unique.* □

Proof of 2.4.

If not, then

$$X := \{x \in W ; f(x) < x\}$$

is nonempty. Let x_0 be least.

$$f(x_0) < x_0$$

$\xrightarrow{\text{APPLY } f}$ $f(\underbrace{f(x_0)}) < \underbrace{f(x_0)}$

So $f(x_0) \in X$, in contradiction to minimality of x_0. q.e.d.

Proof of 2.5

If $f: W \longrightarrow W$ auto

f^{-1} auto

$f(x) \geq x$

$f^{-1}(x) \geq x$

APPLY f

$x \geq f(x)$

$x = f(x)$.

Proof of 2.6

$f: W_1 \longrightarrow W_2$ iso

$g: W_1 \longrightarrow W_2$

$f^{-1}: W_2 \longrightarrow W_1$

$g^{-1}: W_2 \longrightarrow W_1$

$\mathrm{id} = g^{-1} \circ f: W_1 \longrightarrow W_1$ auto

$\mathrm{id} = f^{-1} \circ g: W_2 \longrightarrow W_2$ auto

by 2.5. $\Longrightarrow f = g$.

We say that $I \subseteq X$ is an <u>initial segment</u> of $(X,<)$

[Let $(X,<)$ be linear order]

if for all x, y

$$x \in I,\ y < x \implies y \in I$$

We say I is <u>proper</u> if $I \neq X$.

<u>Observe</u> If $(W,<)$ is a wellorder, then I is a proper initial segment iff there is $w \in W$ s.t.

$$I = I_w := \{x \in W;\ x < w\}$$

<u>Proof.</u> If I is proper, then $W \setminus I \neq \emptyset$, so let w be the least element of $W \setminus I$. Then $I = I_w$. q.e.d.

Lemma 2.7 No wellorder is isomorphic to a proper initial segment of itself.

Proof. $(W,<)$, $w \in W$, suppose

$$f : \underset{w}{\underset{\cup}{W}} \longrightarrow I_w \quad \text{iso.}$$

$$\Longrightarrow f(w) \in I_w$$

$$\Longrightarrow f(w) < w$$ In contradiction to Lemma 2.4.

q.e.d.

No true for arbitrary linear orders:

$(\mathbb{Q},<)$ rational numbers

$\{q \in \mathbb{Q}, q < 0\} \cong \mathbb{Q}$.

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an initial segment of W_2;*
(iii) *W_2 is isomorphic to an initial segment of W_1.*

FUNDAMENTAL THEOREM FOR WELLORDERS

In preparation:

INDUCTION & RECURSION on wellorders.

INDUCTION

Complete Induction (as in "closure under successor") is something rather special on $\mathbb{N}$; cf. HW#3 (9).)

Theorem If $(W,<)$ is a wellorder and $Z \subseteq W$ s.t. f.a. x if $I_x \subseteq Z$, then $x \in Z$. Then $Z = W$.

Proof If $Z \neq W$, then $W \setminus Z \neq \emptyset$. So let w_0 be least in $W \setminus Z$. [This means: $w_0 \notin Z$ ✓ but $\underbrace{\forall z\ z < w_0 \Rightarrow z \in Z}_{I_{w_0} \subseteq Z}$. → Contrad.

q.e.d.

The Recursion Theorem for Wellorders

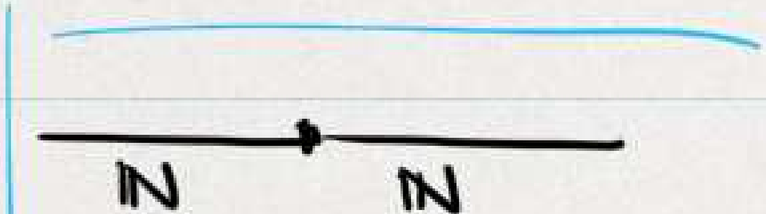
Let $(W,<)$ be a wellorder and φ be a functional total formula. Again, write $F(x)$ for the unique y s.t. $\varphi(x,y,p)$.
Then there is a unique function G with $\mathrm{dom}(G)=W$ and for all $x \in W$

$$(*) \qquad G(x) = F(G \restriction I_x).$$

Proof. (1) Germ: a function g with domain I_x for some $x \in W$ satisfying $(*)$

(2) Germs agree on their common domain

(3) For every $x \in W$, there is a germ g with $x \in \mathrm{dom}(g)$.



(4) Use Replacement on

$$\Phi(x,y,p) :\longleftrightarrow \exists g \text{ germ } g(x)=y$$

to get the range desired, say Y.

(4') Apply Separation to Φ on $W\times Y$. q.e.d.

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an* proper *initial segment of W_2;*
(iii) *W_2 is isomorphic to an* proper *initial segment of W_1.*

FUNDAMENTAL THM FOR WELLORDERS

Wellorders are linearly ordered by their length.

Theorem 2.8. *If W_1 and W_2 are well-ordered sets, then exactly one of the following three cases holds:*

(i) *W_1 is isomorphic to W_2;*
(ii) *W_1 is isomorphic to an initial segment of W_2;*
(iii) *W_2 is isomorphic to an initial segment of W_1.*

<u>Proof.</u> We define a function $G: W_1 \longrightarrow W_2 \cup \{\text{STOP}\}$ by recursion.

$w \in W_1 \quad G(w) := \begin{cases} \text{if } X_w := W_2 \setminus \operatorname{ran}(G \upharpoonright I_w) \neq \emptyset \\ \quad \rightsquigarrow \text{least elt of } X_w. \\ \text{STOP} \quad \text{if } X_w = \emptyset. \end{cases}$

[Note that the range is known in advance, so no need for Replacement.]

By rec. thm., this defines a function $G: W_1 \longrightarrow W_2 \cup \{\text{STOP}\}$.

<u>Case 1.</u> $\text{STOP} \notin \operatorname{ran}(G)$.

By construction $G: W_1 \longrightarrow Z \subseteq W_2$ which is order-preserving & injective.

Again by construction, Z is an initial segment of W_2.

Subcase 1a $Z = W_2$. Then G is an isomorphism from W_1 to W_2, so we're in Case (i) of the thm.

Subcase 1b $Z = I_x$ for $x \in W_2$. Then G is an iso from W_1 to I_x, so we're in Case (ii) of the thm.

Case 2. STOP $\in$ ran(G).

① $\{w \in W_1 ; G(w) \neq \text{STOP}\}$ is an initial segment of W_1.

$Z =$

② If w is least s.t. $G(w) = \text{STOP}$, then $Z = I_w$ and ran$(G \upharpoonright I_w) = W_2$. [before it reaches STOP, it's order-pres. & inj.]

Thus $G \upharpoonright I_w : I_w \longrightarrow W_2$ is an isomorphism. So, we're in case (iii) of the thm.

q.e.d.