Set Theory

LECTURE II

Axioms of Zermelo-Fraenkel

1.1. *Axiom of Extensionality.* *If X and Y have the same elements, then $X = Y$.*

1.2. *Axiom of Pairing.* *For any a and b there exists a set $\{a, b\}$ that contains exactly a and b.*

1.3. *Axiom Schema of Separation.* *If P is a property (with parameter p), then for any X and p there exists a set $Y = \{u \in X : P(u, p)\}$ that contains all those $u \in X$ that have property P.*

1.4. *Axiom of Union.* *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*

1.5. *Axiom of Power Set.* *For any X there exists a set $Y = P(X)$, the set of all subsets of X.*

1.6. *Axiom of Infinity.* *There exists an infinite set.*

1.7. *Axiom Schema of Replacement.* *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x) : x \in X\}$.*

1.8. *Axiom of Regularity.* *Every nonempty set has an $\in$-minimal element.*

1.9. *Axiom of Choice.* *Every family of nonempty sets has a choice function.*

The theory with axioms 1.1–1.8 is the Zermelo-Fraenkel axiomatic set theory ZF; ZFC denotes the theory ZF with the Axiom of Choice.

ZF ZERMELO-FRAENKEL

FST
FINITE SET THEORY

$$\forall x \exists s \forall z \left(z \in s \longleftrightarrow z \in x \wedge \varphi(z) \right)$$

$$\forall p_1 \ldots \forall p_n \forall x \exists s \forall z \left[z \in s \longleftrightarrow z \in x \wedge \varphi(z, p_1, \ldots, p_n) \right]$$

FST : Expansion of language by new relation & function symbol. ← ALLOWED if axioms give existence + uniqueness of the def'd objects

$x \rightsquigarrow \bigcup x$

$x, y \rightsquigarrow \{x, y\}$

$x \rightsquigarrow \{x\}$

$x, y \rightsquigarrow x \cup y$

$x \rightsquigarrow \mathcal{P}(x)$

$\mathcal{P}(x)$.

$Pow(x)$

SEPARATION

$\forall p_1 \forall x \exists s \forall z (z \in s \leftrightarrow z \in x \wedge \varphi(z, p_1))$

INTERSECTION.

$x, y \rightsquigarrow x \cap y$

$p_1 \to y$, $\varphi(z, p_1) \to z \in p_1$

$\forall y \forall x \exists s \forall z (z \in s \leftrightarrow z \in x \wedge z \in y)$

Def. If $G = (V, E)$ is a model, we say that $v \in V$ is AN EMPTY SET if

$G \models \forall z \, (z \notin v)$. [If $\mathrm{pred}_G(v) = \emptyset$.]

Note If $G \models$ Extensionality, then there can be at most one empty set.

Theorem If $G \models$ Separation, then there is an empty set. $G = (V, E)$

Proof. By convention, structures are non-empty, so fix $v \in V$. Consider $\varphi(z) := z \neq z$.

SEP $\forall x \exists s \forall z (z \in s \longleftrightarrow z \in x \wedge \varphi(z))$

($x \to v$, $\varphi(z) \to z \neq z$)

$\exists s \forall z \; z \in s \longleftrightarrow z \in v \wedge z \neq z$ (NEVER TRUE)

$\Longleftrightarrow \exists s \forall z (z \notin s)$

⇝ we are allowed to use $\emptyset$ for the unique empty set (if we're working in FST)

DEF. If $G=(V,E)$ is a model, we call $v\in V$ UNIVERSAL if

$$G\models \forall w\,(w\in v).$$

"the set of all sets"

Theorem If $G\models$ FST, then there is no universal vertex.

["There is no set of all sets."]

Proof SEP $\forall x\,\exists s\,\forall z\,(z\in s \longleftrightarrow z\in x \wedge \varphi(z))$

RUSSELL formula $\varphi(z) := z\notin z$

$$\forall x\,\exists r_x\,\forall z\,(z\in r_x \longleftrightarrow z\in x \wedge z\notin z)$$

what happens if $z=r_x$? $r_x\in r_x \longleftrightarrow r_x\in x \wedge r_x\notin r_x$

$\Longrightarrow r_x\notin x.$

$$\forall x \, \exists r_x \, (r_x \notin x)$$

This means x is not universal.

q.e.d.

FST	NOT YET
$x \rightsquigarrow \bigcup x$ $\quad \emptyset$	$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \ldots$
$x, y \rightsquigarrow \{x, y\}$	
$x \rightsquigarrow \{x\}, \mathcal{P}(x)$	
$x, y \rightsquigarrow x \cup y$	
$x, y \rightsquigarrow x \cap y$	

FINITE SET THEORY

We call a graph G **LOCALLY FINITE** if for all $v \in V$, the set $\mathrm{pred}_G(v)$ is finite.

It's easy to imagine locally finite, but infinite graphs.

G finite $\iff$ there are only finitely many sets

G locally finite $\iff$ there are only finite sets

Theorem There is a locally finite G s.t. $G \models FST$. [Compare G.I.#1 and later.]

While we can't talk about infinity in FST, we can talk about everything else:

⟶ ABSTRACT MATHEMATICS

(Relations, Functions, Structures, Quotient structures)

A binary relation of X is

$$R \subseteq X^2 = X \times X$$

$$= \{(x, x') ; x, x' \in X\}$$

cartesian product

?? what is an ordered pair (x, x')?

We want:

$$(x,y) = (x',y') \iff x = x' \text{ and } y = y'.$$

REQUIREMENT FOR ORDERED PAIR

KURATOWSKI (in FST):

$$(x,y) = \{\{x\},\{x,y\}\}.$$

We observe by basic maths that this satisfies the requirements for ordered pairs.

We have a formula φ

$\varphi(z) \iff z$ is an ordered pair.

GOAL: Define $X \times X = X^2$.

$\{ p \,;\, p$ is an ordered pair $\{\{x\},\{x,y\}\}$ and $x,y \in X \}$

Is this really (provably) a set in FST?

Assume $x,y \in X$.

$$\left.\begin{array}{l}\{x\} \subseteq X \\ \{x,y\} \subseteq X\end{array}\right\} \Longrightarrow \{x\},\{x,y\} \in \mathcal{P}(X)$$

$$\Longrightarrow \{\{x\},\{x,y\}\} \in \mathcal{P}(\mathcal{P}(X))$$

In FST: $X \times X := \{ p \in \mathcal{P}(\mathcal{P}(X)) ; \ \ldots\ldots \}$

The *product* of X and Y is the set of all pairs (x, y) such that $x \in X$ and $y \in Y$:

(1.6) $$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

The notation $\{(x, y) : \dots\}$ in (1.6) is justified because

$$\{(x, y) : \varphi(x, y)\} = \{u : \exists x \exists y\, (u = (x, y) \text{ and } \varphi(x, y))\}.$$

The product $X \times Y$ is a set because

$$X \times Y \subset PP(X \cup Y).$$

An *n-ary relation* R is a set of n-tuples. R is a relation *on* X if $R \subset X^n$. It is customary to write $R(x_1, \dots, x_n)$ instead of

$$(x_1, \dots, x_n) \in R,$$

and in case that R is binary, then we also use

$$x\, R\, y$$

for $(x, y) \in R$.

If R is a binary relation, then the *domain* of R is the set

$$\mathrm{dom}(R) = \{u : \exists v\, (u, v) \in R\},$$

and the *range* of R is the set

$$\mathrm{ran}(R) = \{v : \exists u\, (u, v) \in R\}.$$

Note that $\mathrm{dom}(R)$ and $\mathrm{ran}(R)$ are sets because

$$\mathrm{dom}(R) \subset \bigcup\bigcup R, \qquad \mathrm{ran}(R) \subset \bigcup\bigcup R.$$

The *field* of a relation R is the set $\mathrm{field}(R) = \mathrm{dom}(R) \cup \mathrm{ran}(R)$.

$R \subseteq X \times Y$

(partial)

A binary relation f is a *function* if $(x, y) \in f$ and $(x, z) \in f$ implies $y = z$. The unique y such that $(x, y) \in f$ is the *value* of f at x; we use the standard notation

$$y = f(x)$$

or its variations $f : x \mapsto y$, $y = f_x$, etc. for $(x, y) \in f$.

f is a function *on* X if $X = \mathrm{dom}(f)$. If $\mathrm{dom}(f) = X^n$, then f is an *n-ary function* on X.

f is a function *from* X to Y,

$$f : X \to Y,$$

if $\mathrm{dom}(f) = X$ and $\mathrm{ran}(f) \subset Y$. The set of all functions from X to Y is denoted by Y^X. Note that Y^X is a set:

$$Y^X \subset P(X \times Y).$$

If $Y = \mathrm{ran}(f)$, then f is a function *onto* Y. A function f is *one-to-one* if

$$f(x) = f(y) \quad \text{implies} \quad x = y.$$

injection

surjection

An *n-ary operation on* X is a function $f : X^n \to X$.

The *restriction* of a function f to a set X (usually a subset of $\mathrm{dom}(f)$) is the function

$$f{\restriction}X = \{(x, y) \in f : x \in X\}.$$

A function g is an *extension* of a function f if $g \supset f$, i.e., $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and $g(x) = f(x)$ for all $x \in \mathrm{dom}(f)$.

If f and g are functions such that $\mathrm{ran}(g) \subset \mathrm{dom}(f)$, then the *composition* of f and g is the function $f \circ g$ with domain $\mathrm{dom}(f \circ g) = \mathrm{dom}(g)$ such that $(f \circ g)(x) = f(g(x))$ for all $x \in \mathrm{dom}(g)$.

We denote the *image* of X by f either $f``X$ or $f(X)$:

$$f``X = f(X) = \{y : (\exists x \in X)\, y = f(x)\},$$

and the *inverse image* by

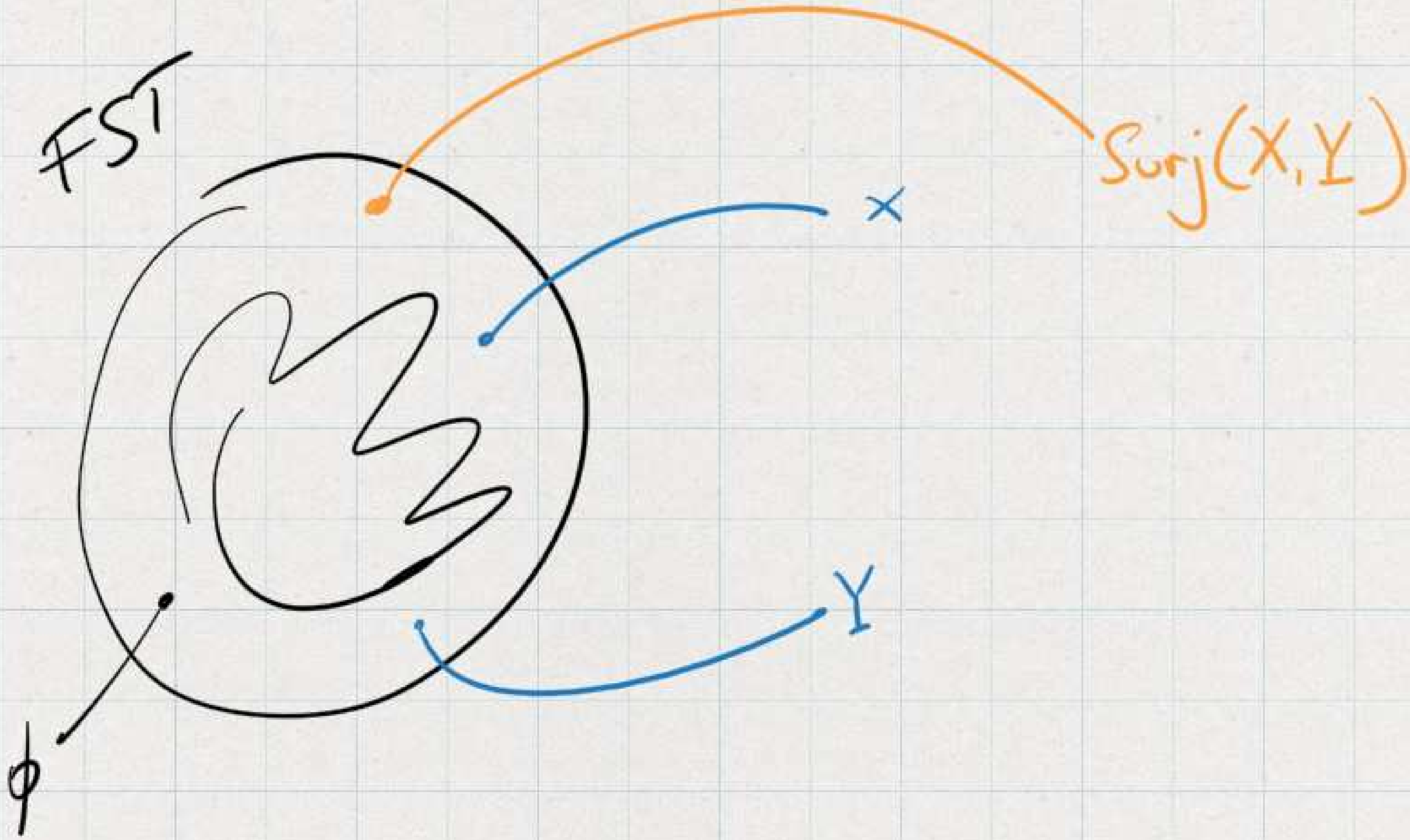
$$f_{-1}(X) = \{x : f(x) \in X\}.$$

If f is one-to-one, then f^{-1} denotes the *inverse* of f:

$$f^{-1}(x) = y \quad \text{if and only if} \quad x = f(y).$$

Y^X is a set by Sep. applied to $P(X \times Y)$.

$P(X \times Y)$ = the set of all relations betw. X & Y


FST
Surj(X,Y)
X
Y
ϕ

An *equivalence relation* on a set X is a binary relation $\equiv$ which is *reflexive*, *symmetric*, and *transitive*: For all $x, y, z \in X$,

$$x \equiv x,$$
$$x \equiv y \text{ implies } y \equiv x,$$
$$\text{if } x \equiv y \text{ and } y \equiv z \text{ then } x \equiv z.$$

A family of sets is *disjoint* if any two of its members are disjoint. A *partition* of a set X is a disjoint family P of nonempty sets such that

$$X = \bigcup\{Y : Y \in P\}.$$

Let $\equiv$ be an equivalence relation on X. For every $x \in X$, let

$$[x] = \{y \in X : y \equiv x\}$$

(the *equivalence class* of x). The set

$$X/\equiv = \{[x] : x \in X\}$$

is a partition of X (the *quotient* of X by $\equiv$). Conversely, each partition P of X defines an equivalence relation on X:

$$x \equiv y \quad \text{if and only if} \quad (\exists Y \in P)(x \in Y \text{ and } y \in Y).$$

$X/_{\equiv} := \{[x]; x \in X\}$

$[x] \subseteq X$

$[x] \in \mathcal{P}(X)$

$X/_{\equiv} := \{z \in \mathcal{P}(X);$ z is an $\equiv$-equivalence class$\}$

Theorem Models of FST cannot be finite.

[It's actually enough to have Sep + Pair or Sep + Power.]

Proof. I am going to show that there is an injection from $\mathbb{N}$ into G.

[Methodological Point: Learn from a proof that uses the informal nat. numbers which properties are really needed for them !!]

Know that G has a unique empty set, say v_0.

Know that for each v there is a vertex $S(v)$ s.t.

SUCCESSOR

$$\forall z \ (z E S(v) \longleftrightarrow z E v \text{ or } z = v)$$

[by Pair & Union in G]

$S(\{\emptyset\}) = \{\emptyset\} \cup \{\{\emptyset\}\}$

$\emptyset \quad S(\emptyset) = \emptyset \cup \{\emptyset\} = \{\emptyset\}$ $= \{\emptyset, \{\emptyset\}\}$

$\emptyset$

$S(\emptyset) = \{\emptyset\}$

$S(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

$S(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\}$

$= \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

$v_0 \in V$ a unique empty set

BY RECURSION $v_{n+1} := S(v_n)$

such that now $f: \mathbb{N} \longrightarrow V$

$f(n) := v_n$.

CLAIM f is an injection.

Main proof method for $\mathbb{N}$ is COMPLETE INDUCTION.

DEF. We call a vertex v TRANSITIVE if for all w, z $w \in z \wedge z \in v \implies w \in v$.

["Set is transitive if elements of elements are elements

EQ: elements are subsets]

TRIVIAL EXAMPLE: $\emptyset$.

OBSERVE If x is transitive, then $S(x)$ is transitive.

$S(x) = x \cup \{x\}$

[$w \in z \in S(x)$

Case 1. $z \in x$: $w \in z \in x \overset{\text{transitivity of } x}{\implies} w \in x. \implies w \in S(x)$.

Case 2. $z = x$: $w \in x \implies w \in S(x)$.]

Claim 1 For all n, $f(n)$ is transitive.

[Proof by ind. $f(0) = v_0$ is transitive since it's the empty set in G.

Suppose $f(n) = v_n$ is transitive, then $f(n+1) = v_{n+1} = S(v_n)$ is transitive by last observation.]

Claim 2 If $n \leq m$, then v_n is a subset of v_m.

[Proof by ind. v_n is a subset of v_n.

v_n is a subset of v_m ⟶ v_n is a subset $v_{m+1} = S(v_m)$
$= v_m \cup \{v_m\}$]

Claim 3 If $n < m$, then $v_n \in v_m$.

[By def. $v_n \in v_{n+1} = S(v_n)$

if $m > n+1$, then by Claim 2 v_m is a superset of v_{n+1}.]

Claim 4 For every n, $v_n \notin v_n$.

[By induction. * $n=0$ Clear, since v_0 has no elements.

Suppose there is some n s.t. $v_n \in v_n$. Let n be least with this property. By *, we know that $n \neq 0$.

So $n = m+1$. $v_m \in v_{m+1} \in v_{m+1} = S(v_m) = v_m \cup \{v_m\}$

INDUCTION PROOF VIA LEAST ELEMENTS

Case 1. $v_{m+1} \in v_m$

transitivity $v_m \in v_{m+1} \in v_m \Longrightarrow v_m \in v_m$

Case 2. $v_{m+1} = v_m$

$v_m \in v_m$

Contradiction to minimality of n.

Claims 3&4 imply that f is injective:

if $n<m$ $f(n)=v_n=v_m=f(m)$

by Cl. 3, $v_n E v_m = v_n$

$\Longrightarrow v_n E v_n$ ↯ Cl. 4. q.e.d.

AXIOM OF INFINITY

Say a set I is **INDUCTIVE** if

$\emptyset \in I$

and if $x \in I$, then $S(x) \in I$.

In FST, there is a formula φ s.t.

I is inductive $\Longleftrightarrow \varphi(I)$

Axiom of Infinity : $\exists x \; \varphi(x)$

where φ is the above formula:

$$\exists x \; \emptyset \in x \wedge \forall y \, (y \in x \longrightarrow S(y) \in x))$$

$$\exists x \; \exists e \; \forall z (z \notin e) \wedge e \in x \;\wedge\; \forall y$$

$$\Big(y \in x \longrightarrow \exists s \, \big(\forall z \, (z \in s \longleftrightarrow z \in y \vee z = y) \\ \wedge \; s \in x \big)\Big)$$

$Z := FST + Inf.$ Zermelo set theory 1908

Ernst Zermelo (1871-1953)

Zermelo, Ernst (1908), "Untersuchungen über die Grundlagen der Mengenlehre I", *Mathematische Annalen*, 65 (2): 261–281.

Zermelo set theory

Z

Abraham Fraenkel (1891-1965)

Axiom of Replacement 1922

Z + Repl.

ZF

Lemma (Z) There is a least inductive set.

[i.e., I inductive and for all J inductive $I \subseteq J$.]

Proof. By Inf., let I be inductive. Define

$$\hat{I} := \{ x \in I \, ; \, \forall J \, (J \text{ is inductive} \longrightarrow x \in J) \}$$

This exists by Sep.

Claim 1 $\hat{I}$ is inductive. $\emptyset \in I$ and for each J $\emptyset \in J \Rightarrow \emptyset \in \hat{I}$.

Suppose $x \in \hat{I} \Rightarrow x \in I$ and for each J, $x \in J$.

$\Downarrow$ $S(x) \in I$ $\qquad$ $\Downarrow$ $S(x) \in J$

$\Longrightarrow S(x) \in \hat{I}$. ✓

Claim 2. If J is inductive, $\hat{I} \subseteq J$.

[By construction.]

Note The definition of $\hat{I}$ did not depend on I.

q.e.d.

In Z, we can define

$$\mathbb{N} := \text{the least inductive set}.$$

[Goal for the rest of today:
Show that this definition is OK]

Look at $\mathbb{N}$:

$\emptyset \in \mathbb{N}$ —— $0 := \emptyset$

$\{\emptyset\} \in \mathbb{N}$ —— $1 := \{\emptyset\}$ $\quad 1 = \{0\}$

$\{\emptyset, \{\emptyset\}\} \in \mathbb{N}$ $\quad 2 := \{\emptyset, \{\emptyset\}\}$ $\quad 2 = \{0, 1\}$

$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \in \mathbb{N}$ $\quad 3 := \ldots$ $\quad 3 = \{0, 1, 2\}$

What properties does $\mathbb{N}$ have?

Theorem (Complete Induction)

For every $Z \subseteq \mathbb{N}$, if $0 = \emptyset \in Z$ and for all x (if $x \in Z$, then $S(x) \in Z$),

then $Z = \mathbb{N}$.

Pf. The assumption mean: Z is inductive, thus $\mathbb{N} \subseteq Z$ (since $\mathbb{N}$ is least ind. set)

$\Rightarrow$ $\mathbb{N} = Z$. q.e.d.

Properties Use the arguments that we did informally using Induction.

① Every $n \in \mathbb{N}$ is transitive. $Z = \{n \in \mathbb{N} ; n \text{ is transitive}\}$

$0 \in Z$, $x \in Z \to S(x) \in Z$.

② Every $n \in \mathbb{N}$, $n \notin n$. $Z = \{n \in \mathbb{N} ; n \notin n\}$

③ For every $n \in \mathbb{N}$, $n = 0$ or $n = S(m)$ for some m.

$Z = \{n \in \mathbb{N} ; n = 0 \text{ or } \exists m \; S(m) = n\}$

④ HOMEWORK For n, m: $n \in m$ or $n = m$ or $m \in n$.

TRICHOTOMOUS

$\Longrightarrow$ That means that

$$n < m :\Longleftrightarrow n \in m$$

Then $(\mathbb{N}, <)$ is a strict total/linear order.

(5) $n \neq m \Longrightarrow S(n) \neq S(m)$.

PEANO STRUCTURE (X, s, x_0)

s.t. $s : X \longrightarrow X$ $\quad \operatorname{ran}(s) = X \setminus \{x_0\}$

s is injective

COMPLETE INDUCTION

$$Z \subseteq X, \ x_0 \in Z \wedge \forall w \, (w \in Z \longrightarrow s(w) \in Z)$$

$$\Longrightarrow Z = X.$$

Theorem Z proves the existence of a Peano structure

$[(\mathbb{N}, S, 0)]$.

Recursion

want to define $G : \mathbb{N} \longrightarrow \mathbb{N}$

given $G(0)$ + fn $F : \mathbb{N} \longrightarrow \mathbb{N}$

$G(n+1) = F(G(n))$

Idea Use Induction to prove recursion.

RECURSION THEOREM (in Z).

Let $F: \mathbb{N} \longrightarrow \mathbb{N}$ be a function and $x_0 \in \mathbb{N}$.
Then there is a unique

$$G: \mathbb{N} \longrightarrow \mathbb{N}$$ satisfying

$n+1 := S(n)$

the RECURSION EQUATIONS

$$(*) \quad \begin{aligned} G(0) &= x_0 \\ G(n+1) &= F(G(n)). \end{aligned}$$

Proof. UNIQUENESS Let G, G' be two functions satisfying $(*)$

$$Z = \{n ; G(n) = G'(n)\}$$ If $Z = \mathbb{N}$, then $G = G'$.

N.T.S Z is inductive. $0 \in Z$, since $G(0) = x_0 = G'(0)$.

If $n \in Z$, $G(n) = G'(n) \Longrightarrow \underset{G(n+1)}{\underbrace{F(G(n))}} = F(G'(n)) = G'(n+1)$. ✓

<u>Idea</u> "Attempts" or "germs".

We call a function g a GERM if $\mathrm{dom}(g) = n \in \mathbb{N}$ and it satisfies (*) on its domain.

E.g., $\emptyset$ [empty function]: $\mathrm{dom}(\emptyset) = \emptyset \in \mathbb{N}$

$g = \{(0, x_0)\}$ is a germ: $\mathrm{dom}(g) = 1 = \{0\} \in \mathbb{N}$

$g(0) = x_0$ (*) ✓

→ $0 \in Z$ of Claim 3

<u>Claim 2</u> If g, g' are germs and $n \in \mathrm{dom}(g) \cap \mathrm{dom}(g')$, then $g(n) = g'(n)$.

[Proof like Claim 1.]

Claim 3 For every $n \in \mathbb{N}$ there is a germ $\underset{\triangledown}{g}$ s.t. $n \in \mathrm{dom}(\underset{\triangledown}{g})$.

[$Z := \{ n \,;\, \exists \underset{\triangledown}{g} \;\; n \in \mathrm{dom}(\underset{\triangledown}{g}) \}$. Claim Z is inductive.

$0 \in Z$ by the orange bit last page

Suppose $n \in Z$, let $\underset{\triangledown}{g}$ be the witness for $n \in Z$.

$$\underset{\triangledown}{g}' := \underset{\triangledown}{g} \cup \{ (n+1, F(\underset{\triangledown}{g}(n))) \}$$

Then $\underset{\triangledown}{g}'$ is a germ and $n+1 \in \mathrm{dom}(\underset{\triangledown}{g}')$.]

Define $G = \{(n,m) ; \exists g \text{ germ with } n \in \text{dom}(g) \text{ and } m = g(n)\}$

$\Phi(n,m)$

IN THE LANGUAGE OF SET THEORY

SEPARATION applied to $\mathbb{N} \times \mathbb{N}$ with formula $\Phi(n,m)$.