Set Theory (MasterMath)

Rudiments of Axiomatic Set Theory (MSc Logic)

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Lecture I

What is set theory?

PRAGMATIC ① LANGUAGE FOR MATHEMATICS

[Conceptual] Distinction between

x and $\{x\}$

$\mathbb{N} \neq \{\mathbb{N}\}$

"belong to" ⇝ $\in / \subseteq$

PHILOSOPHICAL ② FOUNDATIONS OF MATHEMATICS

⟶ single language for all of maths

MATHEMATICAL ③ Particular sets/concepts that become the object of mathematical study:

CARDINAL / ORDINAL

1 & 3 do not require (a priori) any logical techniques.

By 2, set theory is related to the incompleteness phenomenon.

⟶ MODERN SET THEORY (after 1960s)

independence results and comparison of logical strength of set theories.

CARDINALS & ORDINALS

CARDINALS.

BOLZANO (1781–1848)

"Paradoxes of the Infinite"

X, Y sets $\quad X \sim Y$ **EQUINUMEROUS**

if there is a bij. $f : X \longrightarrow Y$.

$2\mathbb{N} \subsetneq \mathbb{N}$; $2\mathbb{N} := \{2n \,;\, n \in \mathbb{N}\}$

$f : n \longmapsto 2n$ is a bijection from $\mathbb{N}$ to $2\mathbb{N}$,

so $2\mathbb{N}, \mathbb{N}$ are equinumerous.

$2\mathbb{N}$ is "strictly smaller than $\mathbb{N}$"

Georg Cantor (1845 - 1918)

"it's a feature, not a bug"

This is a defining feature of infinity. For infinite sets, the notion of equinumerosity $X \sim Y$ is the fundamental notion of size.

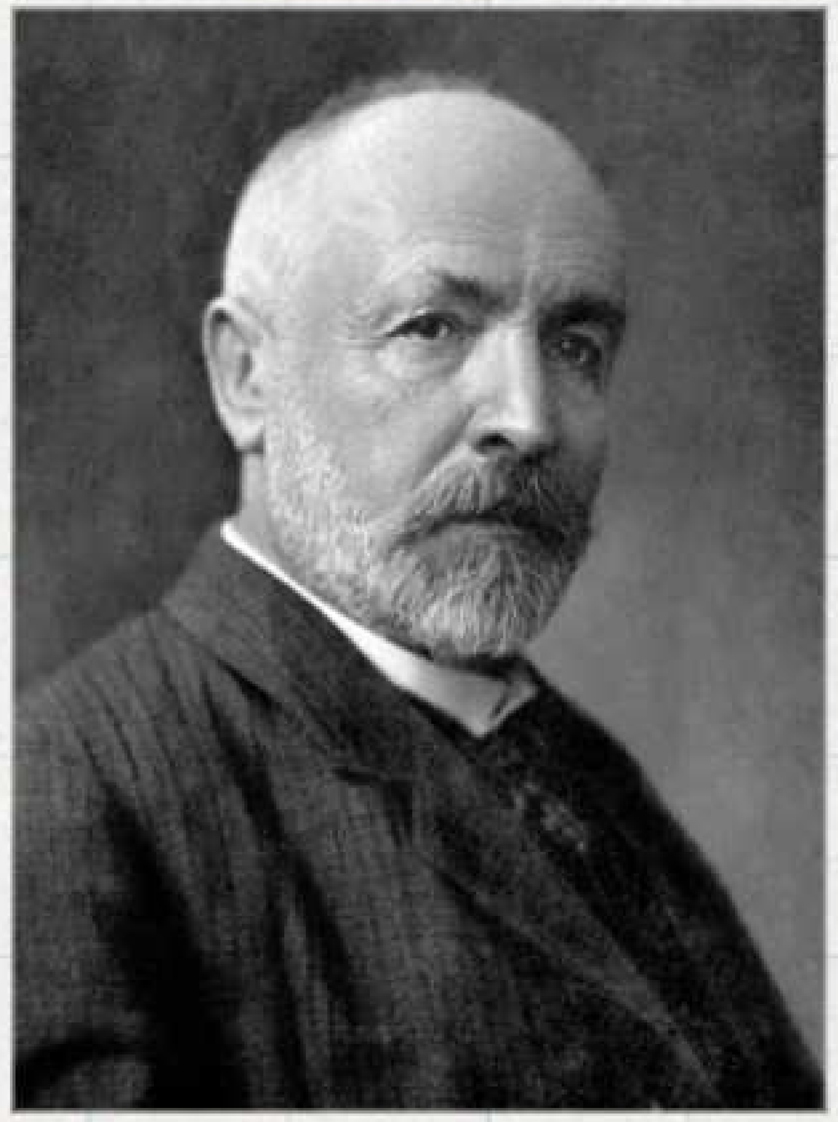
The theory of CARDINALS is thus essentially the theory of sets modulo $\sim$.

Cantor: $\mathbb{N} \sim \mathbb{Q}$.

$\boxed{\mathbb{N} \not\sim \mathbb{R}}$.

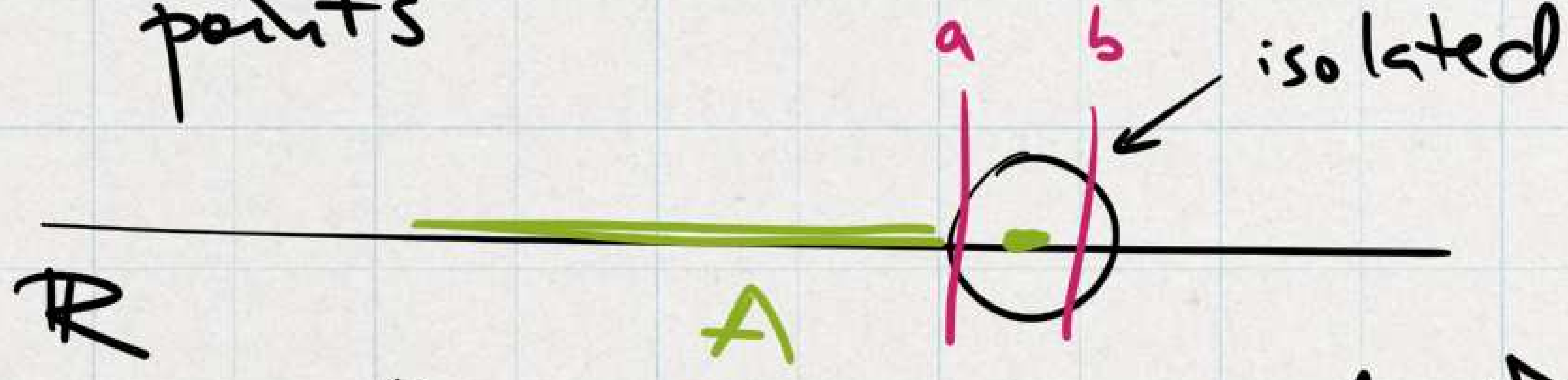
COUNTABILITY

ORDINALS.



Georg Cantor.

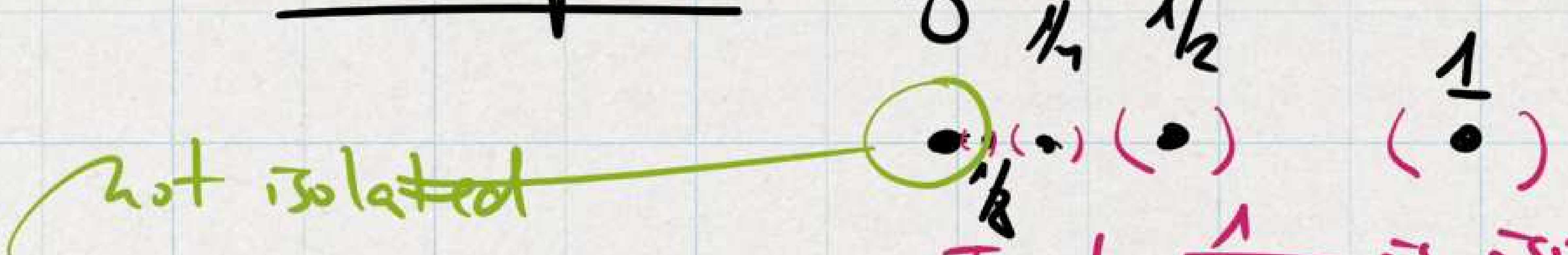
$A \subseteq \mathbb{R}$ **perfect** if it has no isolated points



[$x \in A$ is isolated in A if there are a, b s.t. $(a,b) \cap A = \{x\}$].

Example

$$A = \{0\} \cup \left\{ \frac{1}{2^n} \, ; \, n \in \mathbb{N} \right\}$$



Each $\frac{1}{2^n}$ is isolated;

Theorem (Cantor-Bendixson)

Any closed subset of $\mathbb{R}$ is either countable or contains a non-empty perfect set.

Idea of Proof REMOVE ISOLATED PTS.

There can be at most ctbly many isolated pts, so if

$$A = A' \cup \{x \in A;\ x \text{ isolated}\}.$$

non-isolated

Problem: A' may ~~still~~ have (new) isolated pts.

Idea: ITERATE!

$$A = \{0\} \cup \left\{\frac{1}{2^n};\ n \in \mathbb{N}\right\} \rightsquigarrow A' = \{0\}.$$

$$A_0 := A$$

$$A_{n+1} := (A_n)'$$ Cantor-Bendixson derivative

$$A_\infty := \bigcap_{n \in \mathbb{N}} A_n$$

Problem It might be that $\bigcap_{n \in \mathbb{N}} A_n$ is still not perfect.

Idea: Continue after ∞

$$A_{\infty+1} := (A_\infty)'$$

$$A_{\infty+n+1} := (A_{\infty+n})'$$

$$A_{\infty+\infty} := \bigcap_{n \in \mathbb{N}} A_{\infty+n}$$

Might still be imperfect.

ORDINALS: Indices for this transfinite process.

Relating ordinals & cardinals to each other:

Theorem (Hartogs). There is an uncountable ordinal.

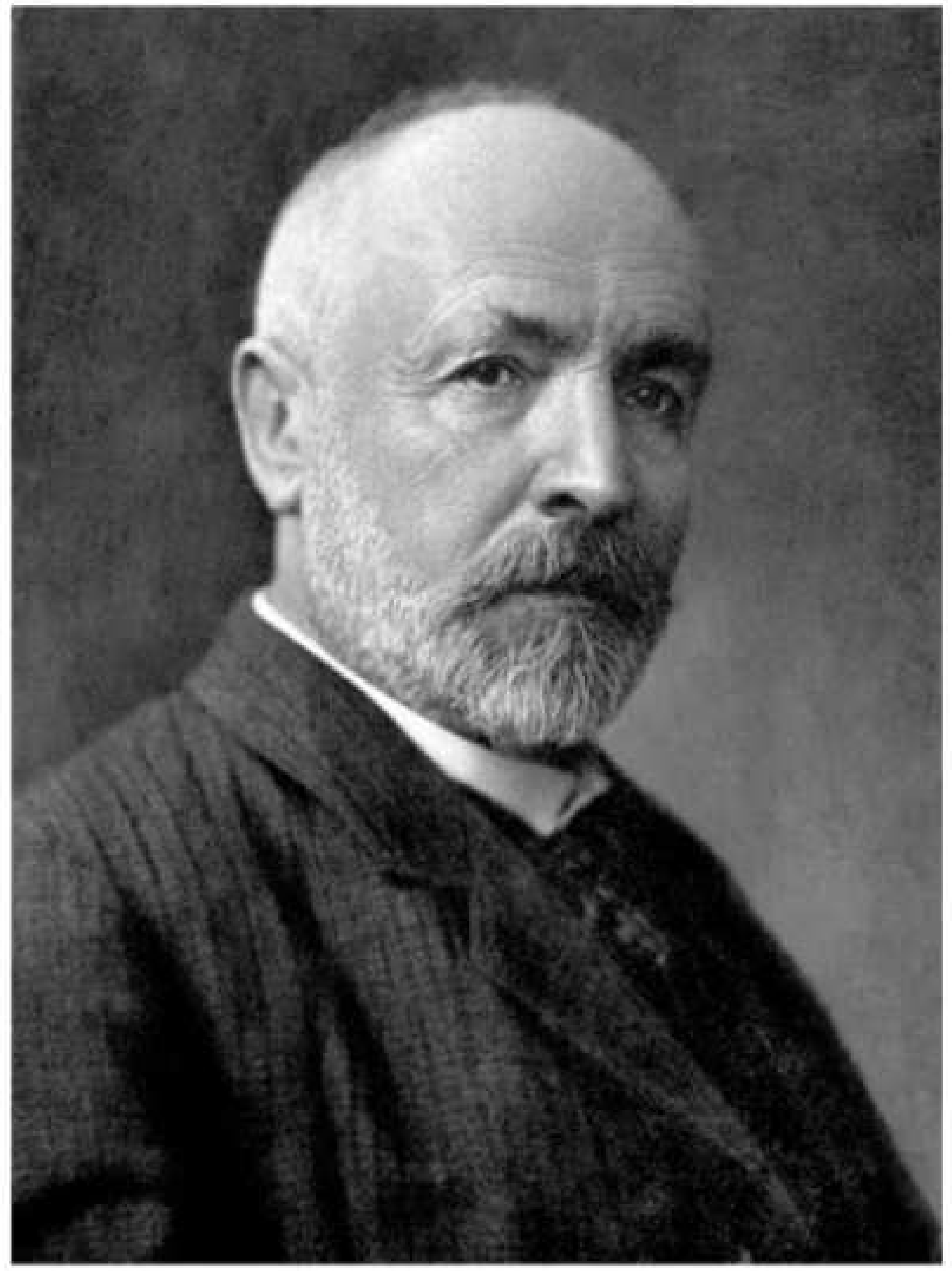
$$\alpha \not\sim \mathbb{N}$$

WONDER: $\alpha \sim \mathbb{R}$

[If there is $\underset{\text{ordinal}}{\alpha} \sim \mathbb{R}$, then this implies consequences for $\mathbb{R}$.

$\rightsquigarrow$ AXIOM OF CHOICE.

What is a set?



Georg Cantor (1845-1918)

Unter einer „Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die „Elemente" von M genannt werden) zu einem Ganzen.

set

By an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

Ὅροι.

α′. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.
β′. Γραμμὴ δὲ μῆκος ἀπλατές.
γ′. Γραμμῆς δὲ πέρατα σημεῖα.
δ′. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾽ ἑαυτῆς σημείοις κεῖται.

Definitions

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.

Hilbert's proposal for geometry was:

IGNORE THE "NATURAL LANGUAGE" DEFINITIONS.

$\mathcal{P}$ points

L lines

$\underline{I} \subseteq P \times L$

"incidence"

$p\underline{I}\ell$ "p lies ~~in~~ on ℓ"

points, lines, planes

tables chair beermugs

Axioms

$$\forall p, q \in P \left(p \neq q \longrightarrow \exists!\, \ell \in L \; p\underline{I}\ell \wedge q\underline{I}\ell\right)$$

$$\forall \ell, \ell' \in L \left(\ell = \ell' \text{ or } \forall p \; \neg(p\underline{I}\ell \wedge p\underline{I}\ell') \text{ or } \exists!\, p \; p\underline{I}\ell \wedge p\underline{I}\ell'\right)$$

⇝ BACK TO SET THEORY

Axioms of set theory determine the behaviour of sets without determining what sets are!

Instead of defining "set", we define the word "model of set theory / universe".

Axioms of Set Theory are not determining what sets are:

! GROUP THEORY define GROUP
RING THEORY define RING
FIELD THEORY define FIELD

SET THEORY defines MODEL OF SET THEORY

Shopping list for my axioms

$x, y \rightsquigarrow x \cup y$

$x, y \rightsquigarrow x \cap y$

$x \rightsquigarrow \{x\}$ SINGLETON

$x, y \rightsquigarrow \{x, y\}$

$x, y, z \rightsquigarrow \{x, y, z\}$

$\vdots$

$x, y \rightsquigarrow x \times y$ Cartesian product

GOAL FOR LECTURES I & II:
Give axioms that allow us to recover all this!

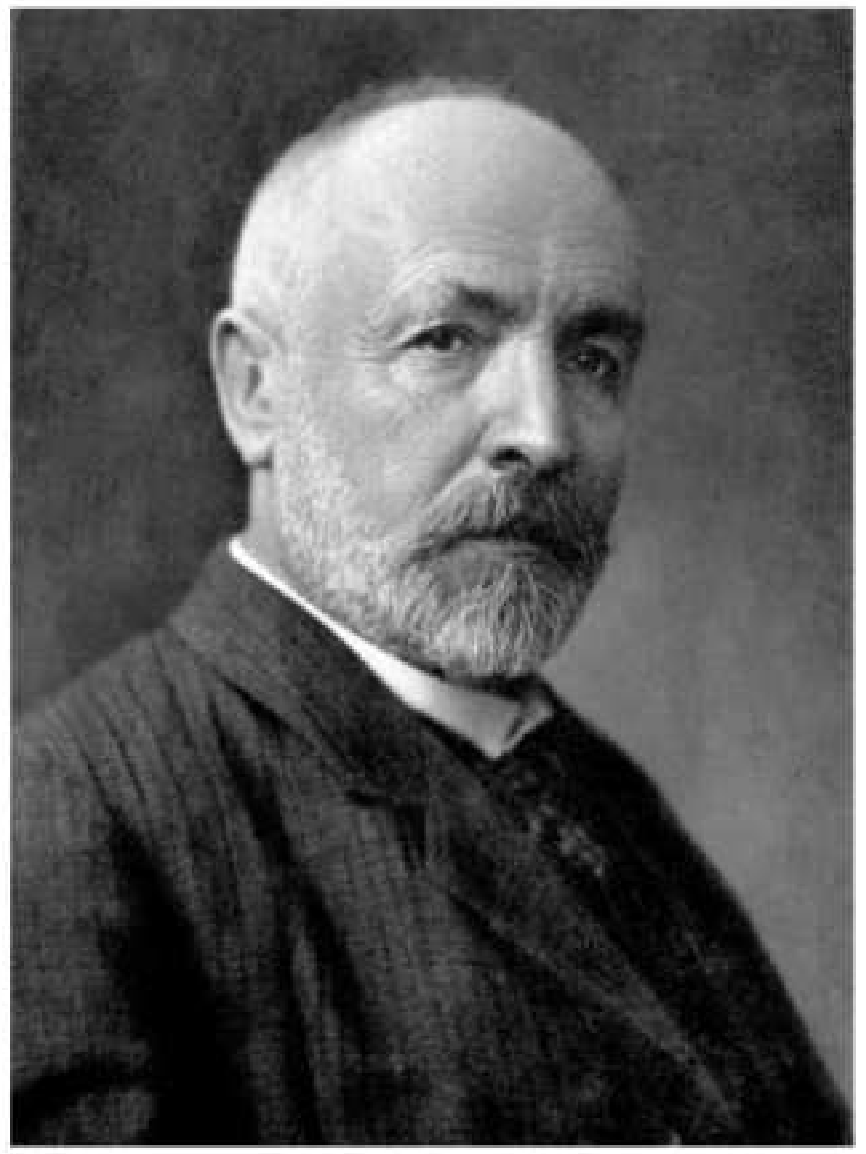
Notions

Relation
Function

Concrete sets

$\emptyset$
$\mathbb{N}$
$\mathbb{Z}$
$\mathbb{Q}$
$\mathbb{R}$
$\mathbb{C}$
etc.

Axioms of Set Theory



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Conceptual background:

the set is determined by its elements

DESCRIPTION DOES NOT MATTER

$\{\text{evening star}\} = \{\text{morning star}\}$

ORDER DOES NOT MATTER

$\{1,2\} = \{2,1\}$

MULTIPLICITY DOES NOT MATTER

$\{1,1,2\} = \{1,2\}$

[MULTISET]

AXIOM OF EXTENSIONALITY

$$\forall x \forall y \left(x=y \longleftrightarrow \underbrace{\left(\forall z (z \in x \longleftrightarrow z \in y)\right)}_{\text{FORMULA DEFINING EQUALITI.}}\right)$$

Formal language $\mathcal{L}_\in$, LST is the first-order language with one binary relation symbol $\in$

"epsilon"
"element of"
"is".

So, models of set theory are just directed graphs

$$G = (V, E)$$

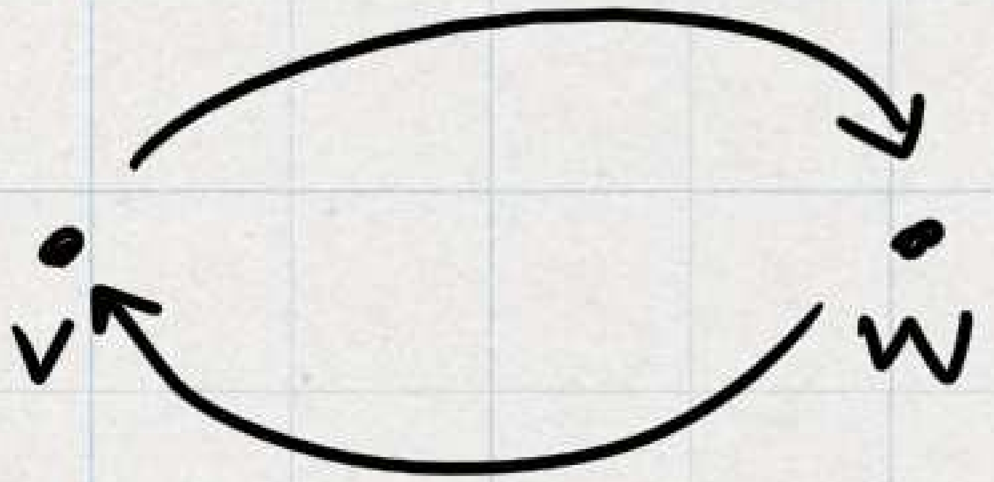
"points"
"vertices"
"sets"

edge relation

$$\underset{v}{\bullet} \longrightarrow \underset{w}{\bullet}$$

$v E w \iff G \models v \in w$

Example



$$G = (V, E) \qquad V = \{v, w\}$$
$$E = \{(v,w), (w,v)\}$$

! CAUTION !

Our notation here is set-theoretic! Isn't that circular?

YES

$$\left.\begin{array}{l} \mathrm{pred}_G(v) = \{w\} \\ \mathrm{pred}_G(w) = \{v\} \end{array}\right\} \longrightarrow$$ so the Axiom of Extensionality is satisfied.

G is called **extensional**!

SIMPLE AXIOMS

Pairing $\forall x \forall y \exists p \; \forall z \, (z \in p \longleftrightarrow z = x \vee z = y)$

$\forall \ldots \quad \exists \ldots$

Union $\forall x \; \exists u \; \forall z \, (z \in u \longleftrightarrow \exists w \, (w \in x \wedge z \in w))$

[Binary union $x, y \longmapsto x \cup y$]

Power set $\forall x \; \exists p \; \forall z \, (z \in p \longleftrightarrow z \subseteq x)$

$\forall w (w \in z \longrightarrow w \in x)$

Simplest examples

$\mathbb{G} = (V, E)$

$V = \{x\}$

$\mathbb{G}_1$

$E = \emptyset$

$\mathbb{G}_2$

$E = \{(x,x)\}$

Which axioms are true in $\mathbb{G}_1$ & $\mathbb{G}_2$?

Extensionality: nothing to show.

Pairing:

$$\forall x \forall y \exists p \forall z (z \in p \longleftrightarrow z = x \vee z = y)$$

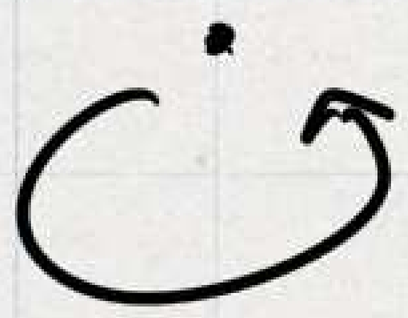
$\Longrightarrow$ there is v with $\mathrm{pred}_{\mathbb{G}}(v) = \{x\}$

$\mathbb{G}_1 \not\models$ Pairing ; $\mathbb{G}_2 \models$ Pairing.

Ext
~~Pair~~
Union
~~Power set~~

G_1

G_2



Ext
Pair
Union
Power set.

__Union__ G_1: x is "the union of x", so $G_1 \models$ Union.

G_2: x is "the union of x", so $G_2 \models$ Union.

$$\forall v\, \exists p\, \forall z\, \big(z \in p \longleftrightarrow \forall w (w \in z \rightarrow w \in v)\big)$$

__Power set__

In G_2, x is "the power set of x".

In G_1, $x \subseteq x$. But no vertex has x as an element, so there is no power set of x.

If $X = \{x\}$ then $\mathcal{P}(X) = \{\emptyset, X\}$

EXPANDING LANGUAGES BY FUNCTION SYMBOLS.

If $\mathcal{L}$ is any language and you want to define a new function symbol f binary (for example) by formula φ then we need existence & uniqueness:

$$\forall x \forall y \exists z \ \varphi(x,y,z) \qquad \text{EXISTENCE}$$

$$\forall x \forall y \forall z \forall z' \ \varphi(x,y,z) \wedge \varphi(x,y,z') \longrightarrow z=z'$$

UNIQUENESS

Extensionality + Pairing allows us to define

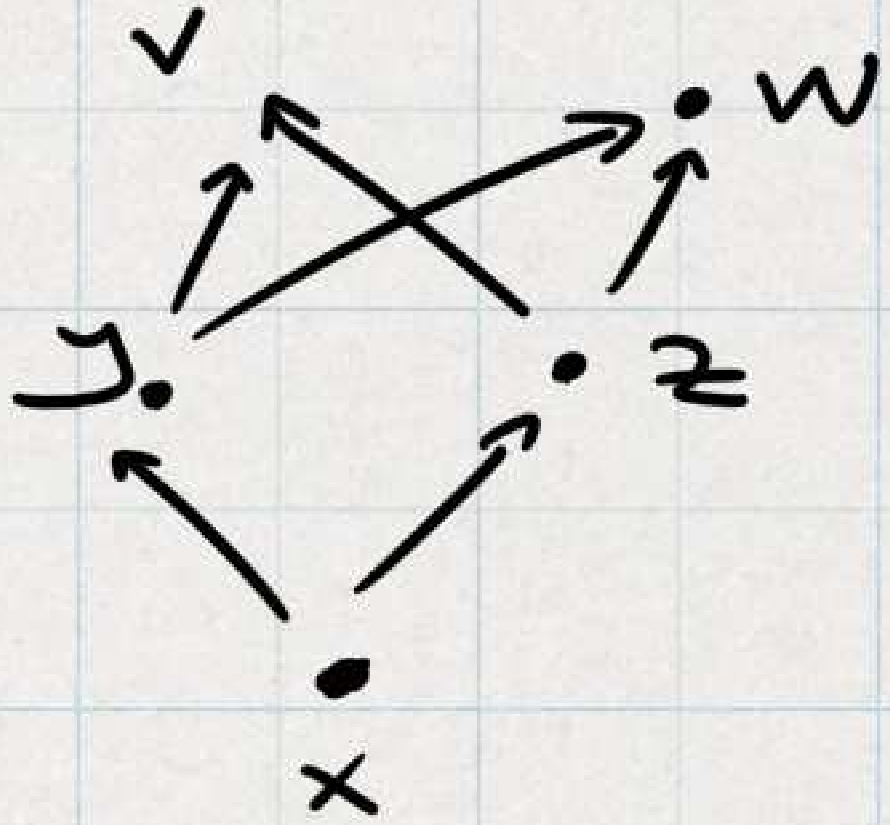
$$x,y \longmapsto \{x,y\}$$

$$x \longmapsto \{x\}$$

Extensionality + Union allows us to define $x \longmapsto \bigcup x$

$z \in \bigcup x \longleftrightarrow \exists v\, (v \in x \wedge z \in v)$

Extensionality + Power Set allows us to define. $x \longmapsto \mathcal{P}(x)$



Notation $\{y, z\}$ makes no sense $\longrightarrow$ MODEL DOES NOT SATISFY EXTENSIONALITY.

Axiom Scheme of Comprehension (*)

Let φ be a formula.

$$\exists c\, \forall z\, (z \in c \longleftrightarrow \varphi(z))$$

WITH PARAMETERS

$$\forall p_1 \dots \forall p_n\, \exists c\, \forall z\, (z \in c \longleftrightarrow \varphi(z, p_1, \dots, p_n))$$

Theorem (Russell 1901)

No directed graph G satisfies (*).

Proof. Take formula $\varphi(z) :\Longleftrightarrow z \notin z$

$\neg(z \in z)$

If (*) holds: $\exists c\, \underbrace{\forall z\, (z \in c \longleftrightarrow z \notin z)}_{\text{instantiate } z \text{ with } c}$

$c \in c \longleftrightarrow c \notin c$. $\lightning$ qed

So, we can't have Comprehension.

SOLUTION ("One step back from disaster")

Axiom Scheme of Separation φ formula

$$\forall p_1 \dots \forall p_n \forall x \exists s \forall z (z \in s \longleftrightarrow z \in x \wedge \varphi(z, p_1, \dots, p_n))$$

same proof as before gives s_x for given x.

$$s_x \in s_x \longleftrightarrow s_x \in x \wedge s_x \notin s_x$$

$\longrightarrow s_x \notin x$.

PREVIEW

Separation $\Longrightarrow$ ex. of $\emptyset$

Separation $\Longrightarrow$ no set of all sets.