Homework Sheet #7

MasterMath: Set Theory
2020/21: 1st Semester
K. P. Hart, Benedikt Löwe, Ezra Schoen, & Ned Wontner

Deadline for Homework Set #7: Monday, 26 October 2020, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(21) Let X be a set; we defined $\mathrm{TC}(X)$, the *transitive closure of* X by recursion. Show that

$$\mathrm{TC}(X) := \bigcap\{T\,;\, T \supseteq X \text{ is a transitive set}\},$$

so $\mathrm{TC}(X)$ is the smallest transitive set containing X.

(22) Let X be a set and $R \subseteq X \times X$ a binary relation. Write $\mathrm{pred}_R(x) := \{y \in X\,;\, y\,R\,x\}$. Recall that we said that R is *well-founded* if every non-empty subset $Y \subseteq X$ has an R-minimal element, i.e., some $y \in Y$ such that $Y \cap \mathrm{pred}_R(y) = \varnothing$. Prove the following *Recursion theorem for well-founded structures*:

Suppose that (X, R) is such that R is a well-founded relation on X. Let Ψ be a functional formula and write $G(v)$ for the unique w such that $\Phi(v, w)$. Then there is a unique function F with $\mathrm{dom}(F) = X$ such that for all $x \in X$, we have that

$$F(x) = G(F{\upharpoonright}\mathrm{pred}_R(x)).$$

(23) Let (X, R) be a well-founded structure such that R is extensional (i.e., if $x \neq y$, then $\mathrm{pred}_R(x) \neq \mathrm{pred}_R(y)$). Show that there is a transitive set T such that $(X, R) \simeq (T, \in)$. Be specific in your proof where you use the assumption of extensionality; what happens if (X, R) is well-founded but not extensional?

[*Hint.* Remember the proof that every well-ordered structure is isomorphic to an ordinal and the recursively defined isomorphism that we used in that proof.]

Show that both the set T and the isomorphism are unique.

(24) Show the following properties of the Mirimanoff rank function:

(a) If $x \in y$, then $\mathrm{rank}(x) < \mathrm{rank}(y)$.

(b) If x is a set, then $\mathrm{rank}(x) = \bigcup\{\mathrm{rank}(y) + 1\,;\, y \in x\}$.

(c) For all ordinals α, we have $\mathbf{V}_\alpha = \{x\,;\, \mathrm{rank}(x) < \alpha\}$.