Homework Sheet #5

MasterMath: Set Theory
2020/21: 1st Semester
K. P. Hart, Benedikt Löwe, Ezra Schoen, & Ned Wontner

Deadline for Homework Set #5: Monday, 12 October 2020, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(15) Suppose α and β are ordinals and define $F(\beta, \alpha) := \{f : \beta \to \alpha\,;\text{ for all but finitely many } \gamma \in \beta,\ f(\gamma) = 0\}$ Define an order $<$ on $F(\beta, \alpha)$ by

$$f < g :\iff f(\mu) < g(\mu) \text{ where } \mu := \max\{\gamma \in \beta\,;\, f(\gamma) \neq g(\gamma)\}.$$

Show that $(F(\beta, \alpha), <) \cong (\alpha^\beta, \in)$.

(16) Prove the *Knaster-Tarski Fixed Point Theorem*: Let X be a set and $F : \wp(X) \to \wp(X)$ a $\subseteq$-monotone function, i.e., if $A \subseteq B$, then $F(A) \subseteq F(B)$. Then F has a fixed point, i.e., a set $A \subseteq X$ such that $A = F(A)$.

Use the Knaster-Tarski Fixed Point Theorem to prove the *Banach Decomposition Theorem*: Let X and Y be sets and $f : X \to Y$ and $g : Y \to X$ arbitrary functions. Then there are disjoint decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $f[X_1] = Y_1$ and $g[Y_2] = X_2$.

[*Hint.* Apply Knaster-Tarski to $F(S) := X \backslash g[Y \backslash f[S]]$.]

Finally, derive the Cantor-Schröder-Bernstein Theorem from the Banach Decomposition Theorem.

(17) Consider the usual topology on the real numbers $\mathbb{R}$: a subset $P \subseteq \mathbb{R}$ is called *open* if for every $x \in P$ there is an $\varepsilon > 0$ such that the open ball around x with radius ε is a subset of P (i.e., $\{y \in \mathbb{R}\,;\, |y - x| < \varepsilon\} \subseteq P$). Let τ be the set of open subsets of $\mathbb{R}$. Prove that there is a bijection between $\mathbb{R}$ and τ.

[*Hint.* Use the fact that $\mathbb{Q}$ is countable and lies dense in $\mathbb{R}$.]