Homework Sheet #4

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #4: Monday, 5 October 2020, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(11) Let S be a set of ordinals. Show that exactly one of the following two cases occurs:

Case 1. S has no largest element, $\bigcup S \notin S$, and $\bigcup S$ is not the successor of any other ordinal;

Case 2. $\bigcup S$ is the largest element of S.

(12) Prove the following properties of ordinal arithmetic (in the following, α, β, and γ are ordinals):

(a) $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

(b) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.

(c) $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$.

(d) $\alpha^{\beta \cdot \gamma} = (\alpha^{\beta})^{\gamma}$.

(13) Prove the following monotonicity properties of ordinal arithmetic (in the following, α, β, and γ are ordinals):

(a) If $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.

(b) If $\alpha < \beta$, then $\gamma + \alpha < \gamma + \beta$.

(c) If $\alpha \leq \beta$, then $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

(d) If $\alpha < \beta$ and $\gamma \neq 0$, then $\gamma \cdot \alpha < \gamma \cdot \beta$.

The strict versions of (a) and (c) do not hold in general: give counterexamples.

(14) Let X be an arbitrary set and $\kappa := \aleph(X) = h(X)$ be the Hartogs aleph of X. Use the proof of Hartogs's theorem to show that there is a surjection from $\mathrm{P}(X \times X)$ onto κ. Deduce that there cannot be an injection from $\mathrm{P}(\mathbb{N})$ into $\mathbb{N}$. Analyse your proof and observe that it does not work for arbitrary sets X (at least not without additional assumptions). Which properties of $\mathbb{N}$ did you use?