Homework Sheet #2

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #3: Monday, 28 September 2020, 2pm. Please hand in via the `elo` webpage as a single pdf file.

(8) We call the following statement the *Empty Set Axiom*: $\exists e \forall z (\neg z \in e)$. Let $\mathbf{G}$ be an arbitrary directed graph. Show that if $\mathbf{G}$ satisfies the Empty Set Axiom and the Axiom Scheme of Replacement, then it satisfies the Axiom Scheme of Separation.

Can you show this without the Empty Set Axiom?

(9) Let $(X, \leq)$ be a linear order with minimal element $0 \in X$ and let $S : X \to X$ be a unary function on X such that for all $x \in X$, we have $x < S(x)$.

- (a) A subset $Z \subseteq X$ is called *S-inductive* if $0 \in Z$ and for all $x \in X$, if $x \in Z$, then $S(x) \in Z$.
- (b) A subset $Z \subseteq X$ is called *order inductive* if for all $x \in X$, if $\{z \in X\,;\, z < x\} \subseteq Z$, then $x \in Z$.
- (c) We say that $(X, \leq, 0, S)$ *satisfies the principle of complete induction* if for every S-inductive set Z, we have that $Z = X$.
- (d) We say that $(X, \leq, 0, S)$ *satisfies the principle of order induction* if for every order inductive set Z, we have that $Z = X$.
- (e) We say that $(X, \leq, 0, S)$ *satisfies the least number principle* if every non-empty subset $Z \subseteq X$ has a $\leq$-least element.

Show that the principle of complete induction implies the principle of order induction and that the principle of order induction and the least number principle are equivalent. Give an example of a structure that satisfies the principle of order induction, but not the principle of complete induction. Give conditions on S under which all three principles are equivalent.

(10) Consider the wellorder $\mathbf{W} = (W, <) := (\mathbb{N}, <) \oplus (\mathbb{N}, <)$. Note that there is a well-defined successor operation on $\mathbf{W}$ defined by $s(i, n) := (i, n + 1)$ for $i \in 2$ and $n \in \mathbb{N}$. Consider the set $D := \{I \subseteq W\,;\, I \text{ is closed under } s\}$. Describe the set D. Consider the partial order $(D, \subseteq)$. Is this a linear order? Is it wellfounded?