Homework Sheet #14

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #14: Monday, 14 December 2020, 2pm.

(46) Let $j : V \to M$ be a non-trivial elementary embedding and let κ be minimal with $j(\kappa) > \kappa$. We showed that κ is measurable because

$$D = \{X \subseteq \kappa : \kappa \in j(X)\}$$

is a κ-complete ultrafilter. This ultrafilter is in fact a normal measure (Jech, top of page 289).

a. The argument in the book takes $X \in D$ and a regressive $f : X \to \kappa$. The claim is that f is constant on a member of D, with value $j(f)(\kappa)$. Give a detailed proof of this.

b. Give an alternative proof by showing, directly from its definition, that D is closed under diagonal intersections.

(47) [Jech, Exercise 17.17] If κ is a successor cardinal, say $\kappa = \lambda^+$, then $\mathcal{L}_{\kappa,\omega}$ does not satisfy the Weak Compactness Theorem.

As in the book use two relations $\prec$ and R (in fact the book seems to assume implicitly that $=$ is part of any language, so formally we have three relations) and constants $\{c_\alpha : \alpha \leqslant \kappa\}$. The intended meaning of $\prec$ is a linear order and R is to code many functions. The set Σ consists of

(1) the axioms for a linear order
(2) the formulas $c_\alpha \prec c_\beta$ for $\alpha < \beta \leqslant \kappa$
(3) a sentence that formulates that a fixed x the relation $R(x, y, z)$ defines z as a function of y; we write $f_x(y) = z$
(4) for all $\alpha \leqslant \kappa$ the sentence φ_α given by $(\forall z)(\exists y)\big(z \prec c_\alpha \to R(c_\alpha, y, z)\big)$
(5) $(\forall x)(\forall y)(\forall z)\big(R(x, y, z) \to \bigvee_{\alpha<\lambda}(y = c_\alpha)\big)$

a. Write down a sentence that accomplishes (3) above

b. Show that (4) and (5) do what the book claims: $\{z : z \prec c_\alpha\} \subseteq \operatorname{ran} f_{c_\alpha}$ and $\operatorname{dom} f_x \subseteq \{c_\alpha : \alpha < \lambda\}$.

c. Prove that every $S \in [\Sigma]^{<\kappa}$ has a model. *Hint*: without loss of generality the set of constants that occur in the sentences in S is of the form $\{c_\kappa\} \cup \{c_\alpha : \alpha < \delta\}$ for some $\delta < \kappa$. Build a model with $\{\kappa\} \cup \delta$ as its universe.

d. Prove that Σ does not have a model. *Hint*: $R(c_\kappa, y, z)$ would code a surjection from λ onto κ.

(48) [Jech, Exercise 17.18] If κ is a singular cardinal then $\mathcal{L}_{\kappa,\omega}$ does not satisfy the Weak Compactness Theorem.

Let A be cofinal in κ and of cardinality less than κ. We use one relation $\prec$ and constants $\{c_\alpha : \alpha \leqslant \kappa\}$. As in the previous exercise $\prec$ is destined to be a linear order. The set Σ consists of

(1) the axioms for a linear order
(2) a sentence that states that $\{c_\alpha : \alpha \in A\}$ is cofinal in this linear order
(3) for every $\alpha < \kappa$ a sentence φ_α that expresses: if $c_\beta \prec c_\kappa$ for all $\beta < \alpha$ then also $c_\alpha \prec c_\kappa$

a. Write down an $\mathcal{L}_{\kappa,\omega}$-sentence that accomplishes (2).

b. Write down an $\mathcal{L}_{\kappa,\omega}$-sentence φ_α that accomplishes (3).

c. Prove that every $S \in [\Sigma]^{<\kappa}$ has a model. *Hint*: the set B of $\alpha \leqslant \kappa$ for which c_α occurs in a sentence in S has cardinality less than κ. Let $\delta = \min \kappa \setminus B$; build a model for S on the set $\kappa + 1$ by inserting κ just before δ

d. Prove that Σ does not have a model. *Hint*: prove that c_κ would become the maximum in the linear order.

(49) [Converse to Jech, Exercise 17.21] Let κ be an uncountable cardinal such that every linearly ordered set of cardinality κ has a well-ordered subset of cardinality κ or an inversely well-ordered subset of cardinality κ. We prove that κ is weakly compact.

a. Prove that κ is not singular. *Hint*: If $\lambda = \operatorname{cf} \kappa < \kappa$ let $\langle \alpha_\eta : \eta < \lambda \rangle$ be increasing, continuous and cofinal in κ, with $\alpha_0 = 0$. Define $\prec$ on κ by

$$\gamma \prec \delta \text{ iff } \begin{cases} \delta < \gamma & \text{if } \gamma, \delta \in [\alpha_\eta, \alpha_{\eta+1}) \text{ for some } \eta \\ \gamma < \delta & \text{otherwise} \end{cases}$$

b. Prove that κ is a strong limit (and hence inaccessible). *Hint*: If $\lambda < \kappa \leqslant 2^{\lambda}$ then apply Exercise (36) from Homework set #10.

c. Prove that κ has the tree property. *Hint*: Assume $(T, <_T)$ be a tree of cardinality κ such that all levels have cardinality less that κ. As in Exercise (40) in Homework set #12 define a linear order $\prec$ on T by first taking a well order $\sqsubset$ of T in type κ and then defining $s \prec t$ if $s <_t t$ or $s_\alpha \sqsubset t_\alpha$, where T_α is the lowest level where s and t have distinct predecessors s_α and t_α respectively.
Let H a subset of T that is well-ordered by $\prec$ (or $\succ$) in order type κ. Prove that in every level of the tree there is exactly one s such that $\{t \in H : s <_T t\}$ has cardinality κ. These points determine a branch.