Homework Sheet #13

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #13: Monday, 7 December 2020, 2pm.

(43) If $\langle A_\alpha : \alpha < \lambda\rangle$ is a sequence of subsets of a set S then a sequence $\langle B_\alpha : \alpha < \lambda\rangle$ of subsets of S is a *flip* of the given sequence if for all α we have $B_\alpha \in \{A_\alpha, S \setminus A_\alpha\}$.

a. Prove: κ is strongly inaccessible iff for every $\lambda < \kappa$ every sequence $\langle A_\alpha : \alpha < \lambda\rangle$ of subsets of κ has a flip $\langle B_\alpha : \alpha < \lambda\rangle$ such that $\bigcap_{\alpha<\lambda} B_\alpha$ has cardinality κ.

b. Prove: κ is weakly compact iff every sequence $\langle A_\alpha : \alpha < \kappa\rangle$ of subsets of κ has a flip $\langle B_\alpha : \alpha < \kappa\rangle$ such that for all $\lambda < \kappa$ the intersection $\bigcap_{\alpha<\lambda} B_\alpha$ has cardinality κ.

c. Prove: κ is measurable compact iff for ever cardinal λ every sequence $\langle A_\alpha : \alpha < \lambda\rangle$ of subsets of κ has a flip $\langle B_\alpha : \alpha < \lambda\rangle$ such that for every subset A of λ of cardinality less than κ the intersection $\bigcap_{\alpha\in A} B_\alpha$ has cardinality κ.

In the following two exercises κ is a measurable cardinal and U is a κ-complete ultrafilter on κ. We let $\mathrm{Ult}_U(V)$ be the corresponding ultrapower, which we identify with its transitive collapse M, and $j = j_U : V \to M$ is the elementary embedding.

(44) This exercise shows directly that $(\kappa^+)^M = \kappa^+$. Just to remind ourselves: $(\kappa^+)^M$ is the smallest ordinal such that there is no surjection $f : \kappa \to (\kappa^+)^M$ *that is in* M; whereas κ^+ is the smallest ordinal such that there is no surjection $f : \kappa \to \kappa^+$ *at all in* V.

a. Show that $(\kappa^+)^M \leqslant \kappa^+$.

b. Prove: if $\alpha < \kappa^+$ then $\alpha < (\kappa^+)^M$. *Hint*: let $f : \kappa \to \alpha$ be a surjection; give an explicit construction of a map $F : \kappa \to V$ such that $f = [F]_U$.

(45) We look at the images of V_κ and $V_{\kappa+1}$.

a. Prove that $j(x) = x$ for all $x \in V_\kappa$.

b. Prove that $(V_\kappa)^M = V_\kappa = j[V_\kappa]$.

c. Prove that $j(V_\kappa) \neq j[V_\kappa]$.

d. Prove that $(V_{\kappa+1})^M = V_{\kappa+1}$.

e. Prove that $V_{\kappa+1} \neq j[V_{\kappa+1}]$, that $V_{\kappa+1} \neq j(V_{\kappa+1})$, and that $j(V_{\kappa+1}) \neq j[V_{\kappa+1}]$.