Homework Sheet #11

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #11: Monday, 23 November 2020, 2pm.

(37) Let κ be an infinite cardinal and let $\prec$ be some well-order of κ. Prove that there is a subset H of κ of cardinality κ such that $\prec$ and $\in$ agree on H, that is: for α and β in H we have $\alpha \in \beta$ iff $\alpha \prec \beta$.

(38) This exercise provides an alternative proof of the instance $\aleph_0 \to (\aleph_0)^2_2$ of Ramsey's Theorem, based on Kőnig's Infinity Lemma.
Let $F : [\omega]^2 \to 2$ be a colouring and let S be the binary tree $2^{<\omega}$. Define subsets A_s of ω, for $s \in S$, by recursion: $A_\emptyset = \omega$, and given A_s define $A_{s,0}$ and $A_{s,1}$ as follows: if $A_s \neq \emptyset$ set $n_s = \min A_s$ and let $A_{s,i} = \{m \in A_s : m > n_s \text{ and } F(\{n_s, m\}) = i\}$; if $A_s = \emptyset$ let $n_s = 0$ and $A_{s,0} = A_{s,1} = \emptyset$. Let $T = \{s \in S : A_s \neq \emptyset\}$.

a. Show that T is downward closed: if $t \in T$ and $s < t$ then $s \in T$.
b. Prove that T is infinite, with finite levels.
c. Let B be an infinite branch of T. Prove that $P = \{n_s : s \in B\}$ has the property that $F(\{k, l\}) = F(\{k, m\})$ whenever $k < l < m$ in P.
d. Show that P has an infinite subset H that is homogeneous for F.

Note: one can prove all instances of Ramsey's theorem from Kőnig's Lemma, using somewhat more complicated trees.

(39) This exercise provides an alternative construction of an Aronszajn tree.
For subsets A and B of $\mathbb{N}$ we say that A is almost contained in B, and we write $A \subseteq^* B$, when $A \setminus B$ is finite.
If s and t are maps from some countable ordinal α to $\mathbb{N}$ then we say that s and t are almost equal, and we write $s =^* t$, when $\{\beta \in \alpha : s(\beta) \neq t(\beta)\}$ is finite.

a. Let $\langle A_n \rangle_n$ be a sequence of infinite subsets of $\mathbb{N}$ such that $A_{n+1} \subseteq^* A_n$ for all n. Prove that there is an infinite set A such that $A \subseteq^* A_n$ for all n.
b. Let $\langle \alpha_n \rangle_n$ be an increasing sequence of countable ordinals and for each n let $s_n : \alpha_n \to \mathbb{N}$ be injective. Assume that $\mathbb{N} \setminus \operatorname{ran} s_n$ is infinite for all n and that $s_m =^* s_n \restriction \alpha_m$ whenever $m < n$. Let $\alpha = \sup_n \alpha_n$. Construct an injective $s : \alpha \to \mathbb{N}$ such that $s_n =^* s \restriction \alpha_n$ for all n *and* $\mathbb{N} \setminus \operatorname{ran} s$ is infinite.
c. Construct a sequence $\langle s_\alpha : \alpha < \omega_1 \rangle$ of functions where $s_\alpha : \alpha \to \mathbb{N}$ is injective for all α *and* $s_\beta =^* s_\alpha \restriction \beta$ whenever $\beta < \alpha$.
d. For $\alpha < \omega_1$ let $T_\alpha = \{t \in \mathbb{N}^\alpha : t =^* s_\alpha\}$. Prove that $\bigcup_{\alpha < \omega_1} T_\alpha$, ordered by $\subset$, is an Aronszajn tree.