Homework Sheet #10

MasterMath: Set Theory
2020/21: 1st Semester
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Deadline for Homework Set #10: Monday, 16 November 2020, 2pm.

(33) Some applications of Ramsey's theorem.

a. Let $\langle L, <\rangle$ be an infinite linearly ordered set. Prove that L has an infinite subset X that is well-ordered by $<$ or an infinite subset Y that is well-ordered by $>$.

b. Prove that every bounded sequence of real numbers has a convergent subsequence (the Bolzano-Weierstraß theorem). *Hint*: Find a monotone subsequence.

c. Let $\langle P, <\rangle$ be an infinite partially ordered set. Prove that P has an infinite subset C that is linearly ordered by $<$ (a chain) or an infinite subset U that is unordered by $<$, which means that if x and y in U are distinct then neither $x < y$ nor $y < x$.

(34) Check the details of Erdős' example that shows $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$: take the open unit interval $(0, 1)$ as the set of cardinality $2^{\aleph_0}$ and define $F(\{x, y\}) = k$ if $2^{-k+1} > |x - y| \geqslant 2^{-k}$. Show there is no three-element homogeneous set.

(35) Prove that $2^\kappa \not\to (3)^2_\kappa$ for every cardinal κ. *Hint*: Take the *set* 2^κ of functions from κ to 2 and define $F : [2^\kappa]^2 \to \kappa$ by $F(\{x, y\}) = \min\{\alpha : x(\alpha) \neq y(\alpha)\}$.

(36) Prove that $2^\kappa \not\to (\kappa^+)^2_2$ for every cardinal κ. *Hint*: Take the *set* 2^κ of functions from κ to 2. Let $<$ be the lexicographic order of 2^κ, defined by $x < y$ iff $x(\delta) < y(\delta)$ where $\delta = \min\{\alpha : x(\alpha) \neq y(\alpha)\}$ and let $\prec$ be some well-order of 2^κ. $F : [2^\kappa]^2 \to 2$ by $F(\{x, y\}) = 1$ if $\prec$ and $<$ agree on $\{x, y\}$ and let $F(\{x, y\}) = 0$ in the opposite case. The crux is to prove that any set well-ordered by $<$ has cardinality at most κ.