Group Interaction #9

MasterMath: Set Theory
2020/21: 1st Semester
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(1) This exercise explores closed and unbounded sets some more and their relation to the set of submodels of a given model and normal filters.
Let κ be a regular cardinal. Remember that a function $f : \kappa \to \kappa$ said to be normal if it is order-preserving (if $\alpha < \beta$ then $f(\alpha) < f(\beta)$) and continuous (if α is a limit ordinal then $f(\alpha) = \sup_{\beta<\alpha} f(\beta)$).
There is a close connection between closed unbounded sets and normal functions.

a. Prove: if f is normal then $f[\kappa]$ is closed and unbounded.

b. Prove: if C is closed and unbounded then there is a normal function f such that $C = f[\kappa]$.

The following two parts show the relation between closed unbounded sets and families of submodels of a given model.

c. Let $f : \kappa^n \to \kappa$ be an arbitrary function for some $n \in \mathbb{N}$. Show that $C_f = \{\alpha : f[\alpha^n] \subseteq \alpha\}$ is closed and unbounded.

d. Let $*$ be a binary operation such that $(\kappa, *)$ is a group. Prove that the set of α such that $(\alpha, *)$ is a subgroup of $(\kappa, *)$ is closed and unbounded.

(2) In class we proved the ultrafilter theorem using Zorn's Lemma.
Here is sketch of a construction based on the well-ordering theorem. (Many constructions of ultrafilters in topology and set theory are by transfinite recursion.)
Let X be a set.

a. Prove that a filter F on X is an ultrafilter if and only if for every function $f : X \to \{0, 1\}$ there is a member of F on which f is constant.

b. Prove: if F is a filter and $f : X \to \{0, 1\}$ is a function then there is a filter G that is finer than F and that has a member on which f is constant.

Now let $\langle f_\alpha : \alpha < 2^\kappa \rangle$, where $\kappa = |X|$, enumerate the set of all functions from X to $\{0, 1\}$.

c. Let F be an arbitrary filter on X; show how to build an increasing chain $\langle F_\alpha : \alpha < 2^\kappa \rangle$ of filters such that $F_0 = F$ and for every α there an element of $F_{\alpha+1}$ on which f_α is constant. Deduce that $U = \bigcup_{\alpha<2^\kappa} F_\alpha$ is an ultrafilter.

d. The construction in the previous part requires a choice at every successor step. How would you use the well-ordering theorem to make that choice explicit?