Group Interaction #8

MasterMath: Set Theory
2020/21: 1st Semester
K. P. Hart, Benedikt Löwe, Ezra Shoen, & Ned Wontner

In class we saw the statement

> An infinite cardinal κ is singular if and only if there exist a cardinal $\lambda < \kappa$ and a family $\{S_\alpha : \alpha < \lambda\}$ of subsets of κ such that $|S_\alpha| < \kappa$ for all α and $\kappa = \bigcup_{\alpha<\lambda} S_\alpha$.
>
> The smallest such cardinal λ is equal to $\operatorname{cf}\kappa$.

(1) Prove the equivalence. Check whether your proof uses the Axiom of Choice. In fact, neither of the implications needs AC, so try to come up with a proof in ZF.

(2) Consider variations of the statement. Can the S_α be taken disjoint? Can the cardinalities of the S_α be all singular? All regular?

The Singular Cardinals Hypothesis (SCH) was introduced in class:

> For every singular cardinal κ: if $2^{\operatorname{cf}\kappa} < \kappa$ then $\kappa^{\operatorname{cf}\kappa} = \kappa^+$.

(3) Prove that the Generalized Continuum Hypothesis implies SCH.

(4) Theorem 5.22 in Jech describes the behaviour of the continuum function and cardinal exponentiation under the assumption of SCH.

> **Theorem 5.22.** *Assume that* SCH *holds.*
>
> (i) *If κ is a singular cardinal then*
> (a) $2^\kappa = 2^{<\kappa}$ *if the continuum function is eventually constant below κ,*
> (b) $2^\kappa = (2^{<\kappa})^+$ *otherwise.*
>
> (ii) *If κ and λ are infinite cardinals, then:*
> (a) *If $\kappa \le 2^\lambda$ then $\kappa^\lambda = 2^\lambda$.*
> (b) *If $2^\lambda < \kappa$ and $\lambda < \operatorname{cf}\kappa$ then $\kappa^\lambda = \kappa$.*
> (c) *If $2^\lambda < \kappa$ and $\operatorname{cf}\kappa \le \lambda$ then $\kappa^\lambda = \kappa^+$.*

It is prefaced by the statement that cardinal exponentiation is determined by the continuum function on regular cardinals. Describe the/an algorithm that allows you to calculate κ^λ for arbitrary κ and λ under the assumption that you have an oracle that tells you the value of 2^μ for *regular* μ, that oracle could be in the form of a (class) function F from the class of ordinals to itself such that $2^{\aleph_\alpha} = \aleph_{F(\alpha)}$ whenever $\aleph_\alpha$ is regular.

(5) Use your algorithm to determine the following powers, under the assumption that SCH holds and that $2^{\aleph_{\alpha+1}} = \aleph_{\omega+\alpha^2+1}$ for all α.

a. $\aleph_\omega^{\aleph_\omega}$

b. $\aleph_{\omega+2}^{\aleph_2}$

c. $\aleph_{\omega_1\cdot 2}^{\aleph_1}$

d. $\aleph_5^{\aleph_{\omega_1}}$