Group Interaction #7

MasterMath: Set Theory
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In the seventh group interaction, we explore levels of the von Neumann hierarchy as models of set theory.

(1) In class, we stated that if λ is a limit ordinal, then $\mathbf{V}_\lambda$ is a model of FST. Go through all of the axioms carefully and understand why this claim is true.

(2) In class, we sketched the argument that neither $\mathbf{V}_{\omega+\omega}$ nor $\mathbf{V}_{\omega_1}$ satisfy the axiom scheme of Replacement. Give a careful argument for this claim by specifying precisely the functional formula Φ for which the scheme is violated in the corresponding models.

(3) Generalise the argument from (2) to show the following claims:

(a) If κ is a cardinal such that there is a $\lambda < \kappa$ and a cofinal function $f : \lambda \to \kappa$ that is definable in $\mathbf{V}_\kappa$ (i.e., there is a formula Φ such that $f(\alpha) = \beta$ if and only if $\mathbf{V}_\kappa \models \Phi(\alpha, \beta)$), then $\mathbf{V}_\kappa$ cannot be a model of the axiom scheme of Replacement.

(b) If κ is a cardinal such that there is a $\lambda < \kappa$ and a surjection $g : \mathrm{P}(\lambda) \to \kappa$ that is definable in $\mathbf{V}_\kappa$ (i.e., there is a formula Ψ such that $g(X) = \alpha$ if and only if $\mathbf{V}_\kappa \models \Psi(X, \alpha)$), then $\mathbf{V}_\kappa$ cannot be a model of the axiom scheme of Replacement.

Explain why the assumption of *definability* was needed in your argument.

(4) Let $X \subseteq \mathbf{V}_\omega$. We say that X is *closed under pairing* if for any $x, y \in X$, also $\{x, y\} \in X$. We say that X is *closed under union* if for any $x \in X$, also $\bigcup x \in X$. Characterise the subsets of $\mathbf{V}_\omega$ that are closed under both pairing and union.

(5) Remember the graph model constructions that we did in the first *Group Interaction*. We had *augmentation operations* that took an extensional graph $\mathbf{G} = (V, E)$ and extended it to a bigger extensional graph $\mathbf{G}' = (V', E')$ such that no new incoming edges for old vertices were constructed (i.e., if $v \in V$ and $(w, v) \in E'$, then $(w, v) \in E$; in other words, $\mathbf{G}'$ is an *end extension* of $\mathbf{G}$).

(Now is a good moment to go back to the task sheet of *Group Interaction #1* and re-familiarise yourself with the notions of *Pairing Closure* and *Power Set Closure*, as well as the notions of an *extensional graph* and a *locally finite graph*.)

Give conditions on the *augmentation operation* $\mathbf{G} \mapsto \mathbf{G}'$ that imply that if $\mathbf{G}$ is isomorphic to a subset of $\mathbf{V}_\omega$, then $\mathbf{G}'$ is isomorphic to a subset of $\mathbf{V}_\omega$.

(6) Using (4) and (5), formulate and prove a result that shows that there can be no graph model construction starting from the single irreflexive point such that the resulting model is a model of the pairing and the union axiom, but not of the power set axiom.

[Note that "formulating the result" means that you need to specify precisely what you mean by *graph model construction*. You need to do this specification in such a way that you can prove the claim.]