Group Interaction #6

MasterMath: Set Theory
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In the sixth group interaction, we shall look at equivalents of the Axiom of Choice, in particular *Zorn's Lemma*. The following is an excerpt from Jech's book (p. 49):

> In algebra and point set topology, one often uses the following version of the Axiom of Choice. We recall that if $(P, <)$ is a partially ordered set, then $a \in P$ is called *maximal* in P if there is no $x \in P$ such that $a < x$. If X is a nonempty subset of P, then $c \in P$ is an *upper bound* of X if $x \leq c$ for every $x \in X$.
>
> We say that a nonempty $C \subset P$ is a *chain* in P if C is linearly ordered by $<$.
>
> **Theorem 5.4 (Zorn's Lemma).** *If $(P, <)$ is a nonempty partially ordered set such that every chain in P has an upper bound, then P has a maximal element.*

(1) Prove Zorn's Lemma as stated above in the theory ZFC. Be very explicit about your use of the Axiom of Choice in your proof, i.e., state precisely for which set you need the choice functions.

(2) Zorn's Lemma is equivalent to the Axiom of Choice in the following sense: if ZL is the formula expressing Zorn's Lemma, then ZF + ZL proves AC. Prove that claim.

(3) Remember (from other mathematics classes) typical uses of Zorn's Lemma in mathematics: e.g., the proof that every vector space has a basis or that every field has an algebraic closure. Re-construct these proofs from memory. Now that you know by (1) & (2) that AC and ZL are equivalent, you should be able to prove these results directly with the Axiom of Choice without applying Zorn's Lemma. Re-write your proofs in such a way that they use choice functions directly.

(4) The following statement is called the *order extension principle* OEP: "For every partial order $(X, \leq)$ there is a relation $\leq^*$ extending $\leq$ (i.e., $\leq \subseteq \leq^*$ as subsets of $X \times X$) such that $(X, \leq^*)$ is a linear order". Show that Zorn's Lemma implies OEP.

    There are (at least) two different natural partial orders whose maximal elements are linearly ordered extensions of $(X, \leq)$. Can you give two different examples?

(5) The following statement is called the *dense ordering principle* DOP: "For every infinite set X, there is a relation $\leq$ on X such that $(X, \leq)$ is a dense linear order". Show that the Axiom of Choice implies DOP.

    [*Hint.* A countable set can be densely linearly ordered (since $\mathbb{Q}$ is countable). Use the Axiom of Choice to split the set X into a linearly ordered collection of countable sets and then order each of the countable sets like $\mathbb{Q}$.]

(6) The following statement is called the *(linear) order principle* OP: "For every set X, there is a relation $\leq$ on X such that $(X, \leq)$ is a linear order". Compare the statements AC, OEP, DOP, and OP. Which of them imply which (in the base theory ZF? Can you draw an implication diagram?