Group Interaction #5

MasterMath: Set Theory
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In the fifth group interaction, we shall look at Hessenberg's Theorem. The following is an excerpt from Jech's book (pp. 30–31):

> ### The Canonical Well-Ordering of $\alpha \times \alpha$
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> We define a well-ordering of the class $Ord \times Ord$ of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the *canonical well-ordering* of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class Ord, and we have a one-to-one function Γ of Ord^2 onto Ord. For many α's the order-type of $\alpha \times \alpha$ is α; in particular for those α that are alephs.
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> We define:
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> (3.12) $$(\alpha,\beta) < (\gamma,\delta) \leftrightarrow \begin{array}{l} \text{either } \max\{\alpha,\beta\} < \max\{\gamma,\delta\}, \\ \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\} \text{ and } \alpha < \gamma, \\ \text{or } \max\{\alpha,\beta\} = \max\{\gamma,\delta\},\ \alpha = \gamma \text{ and } \beta < \delta. \end{array}$$
>
> The relation $<$ defined in (3.12) is a linear ordering of the class $Ord \times Ord$. Moreover, if $X \subset Ord \times Ord$ is nonempty, then X has a least element. Also, for each α, $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. If we let
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> $$\Gamma(\alpha,\beta) = \text{the order-type of the set } \{(\xi,\eta) : (\xi,\eta) < (\alpha,\beta)\},$$
>
> then Γ is a one-to-one mapping of Ord^2 onto Ord, and
>
> (3.13) $$(\alpha,\beta) < (\gamma,\delta) \quad \text{if and only if} \quad \Gamma(\alpha,\beta) < \Gamma(\gamma,\delta).$$

(1) Draw a picture of what this function is doing. Calculate the values of $\Gamma(2,2)$, $\Gamma(3,3)$, $\Gamma(4,4)$, $\Gamma(0,\omega)$, $\Gamma(\omega,0)$, $\Gamma(\omega,\omega)$, $\Gamma(\omega,\omega+1)$, and $\Gamma(\omega+3,\omega+3)$.

(2) Check Jech's claims in this text excerpt:

   (a) $<$ is a linear order,

   (b) if X is a nonempty class of pairs of ordinals, then it has a $<$-least element,

   (c) for each α, $\alpha \times \alpha$ is the initial segment of $<$ given by $(0, \alpha)$.

(3) Use the function Γ to prove Hessenberg's Theorem: for any infinite ordinal α, there is a bijection between α and $\alpha \times \alpha$.

   [*Hint.* It's enough to prove this for cardinals $\aleph_\gamma$ (why?). Prove it for these by induction on γ: assume that it is the case for all ordinals $\xi < \gamma$ and show it for γ.]

(4) Use (3) to show that for infinite cardinal numbers $\kappa \leq \lambda$, the following three sets are in bijection with each other: λ, $\kappa \times \lambda$, and the disjoint union of κ and λ (i.e., $\kappa \times \{0\} \cup \lambda \times \{1\}$).