Group Interaction #4

MasterMath: Set Theory
2020/21: 1st Semester
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In the fourth group interaction, we shall look at the *Cantor Normal Form* of ordinals. **You are allowed to use the rules of ordinal arithmetic and monotonicity that you will do on homework sheet #4.**

(0) *Ordinal subtraction.* Read and understand the following argument in detail:

> If $\beta \leq \alpha$, then there is a unique ordinal γ such that $\alpha = \beta + \gamma$.
>
> [Let η be least such that $\beta + \eta > \alpha$. (Why does this have to exist?) Observe that $\eta \neq 0$. Check that η cannot be a limit ordinal: if it is, then for all $\xi < \eta$, we have $\beta + \xi \leq \alpha$, but then $\beta + \eta = \bigcup\{\beta + \xi\,;\, \xi < \eta\} \leq \alpha$. Thus $\eta = \gamma + 1$. Check that $\alpha = \beta + \gamma$: by minimality of η, we have that $\beta + \gamma \leq \alpha < \beta + (\gamma + 1) = \beta + \eta$, thus $\beta + \gamma = \alpha$. Uniqueness follows from Homework (13b).]

Note that the order of addition matters here: if $\alpha = \omega + 1$ and $\beta = \omega$, then there is no γ such that $\alpha = \gamma + \beta$. (Why?)

(1) *Ordinal Division (with remainder).* Let $0 < \beta < \alpha$. Show that there are unique γ and ϱ such that $\alpha = \beta \cdot \gamma + \varrho$ and $\varrho < \beta$.

(2) Observe that the order of multiplication matters here: find examples of ordinals $\beta < \alpha$ such that it is not possible to write α as $\gamma \cdot \beta + \varrho$ with $\varrho < \beta$.

(3) *Ordinal Logarithm (with remainders).* Let $1 < \beta < \alpha$. Show that there are unique γ, ϱ_0, and ϱ_1 such that

- (a) $\alpha = \beta^\gamma \cdot \varrho_0 + \varrho_1$,
- (b) $\varrho_0 < \beta$, and
- (c) $\varrho_1 < \beta^\gamma$.

(4) Let γ be an ordinal. A finite sequence $(\gamma_0, ..., \gamma_n)$ with $\gamma_0 \geq \gamma_1 \geq ... \geq \gamma_n$ is called a *Cantor Normal Form of* γ if

$$\gamma = \omega^{\gamma_0} + \cdots + \omega^{\gamma_n}.$$

Prove that every ordinal $\gamma > 0$ has a unique Cantor Normal Form.

(5) Determine the Cantor Normal Form of the following ordinals:

- (a) $1 + \omega$,
- (b) $2 \cdot \omega$,
- (c) $\omega \cdot 2$,
- (d) $(\omega + 2) \cdot (\omega \cdot 2 + 2)$,
- (e) $(\omega + 2)^{\omega + 2}$, and
- (f) ω_1, the smallest uncountable ordinal.

(6) An ordinal γ is called *selfnormal* if it is its own Cantor Normal Form, i.e., $\omega^\gamma = \gamma$. Can you find a selfnormal ordinal? Can you find a countable selfnormal ordinal?

(7) The Cantor Normal Form from (4) uses the base ω. Let β be any ordinal. Formulate and prove a version of the Cantor Normal Form theorem for the base β. For which ordinals β can you prove the theorem?

($\star$) *Additional food for thought* (there will be no time to do this during the group interaction, but you might find it interesting to think about this independently or in preparation for the exam).

(a) An ordinal γ is called a *gamma number* (or *principal number of addition* if it is closed under addition, i.e., for all $\alpha, \beta \in \gamma$, we have $\alpha + \beta \in \gamma$. Show that $\gamma \neq 0$ is a gamma number if and only if there is a ξ such that $\gamma = \omega^\xi$.

(b) An ordinal δ is called a *delta number* (or *principal number of multiplication* if it is closed under multiplication, i.e., for all $\alpha, \beta \in \gamma$, we have $\alpha \cdot \beta \in \gamma$. Show that $\delta \notin \{0, 1\}$ is a delta number if and only if there is a ξ such that $\delta = \omega^{(\omega^\xi)}$.

(c) An ordinal ε is called an *epsilon number* (or *principal number of exponentiation* if it is closed under exponentiation, i.e., for all $\alpha, \beta \in \gamma$, we have $\alpha^\beta \in \gamma$. Show that $\varepsilon \notin \{0, 2, \omega\}$ is an epsilon number if and only it is selfnormal in the sense of (6).