Group Interaction #3

MasterMath: Set Theory
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K. P. Hart, Benedikt Löwe, Ezra Schoen, & Ned Wontner

In the third group interaction, we shall explore the set-theoretic construction of the integers, the rational numbers, and the real numbers.

A structure $\mathbf{A} = (A, +, 0)$ is called *cancellative abelian monoid* if for all $a, b, c \in A$, we have:

$$\begin{aligned} a + (b + c) &= (a + b) + c, \\ a + b &= b + a, \\ a + 0 &= a, \\ 0 + a &= a \text{ and} \\ \text{if } a + b = a + c&, \text{ then } b = c. \end{aligned}$$

The last condition is called the *cancellation law.* If $\mathbf{A}$ is a cancellative abelian monoid, we define a relation $\approx$ on $A \times A$ by $(a, b) \approx (a', b')$ if and only if $a + b' = b + a'$.

(1) Show that $\approx$ is an equivalence relation. Highlight the use of the cancellation law in your argument and consider what happens if the monoid $\mathbf{A}$ is not cancellative.

(2) Writing $[a, a']_\approx$ for the $\approx$-equivalence class of the pair (a, a'), define an operation $+$ on the set of all $\approx$-equivalence classes $A \times A/\approx$ such that $(A \times A/\approx, +, [0, 0]_\approx)$ is a group and the map
$$i : A \to G : a \mapsto [a, 0]_\approx$$
is a structure-preserving injection.

(3) Using the construction in (2) and applying it to the natural numbers $\mathbb{N}$, argue that models of Zermelo set theory Z contain a structure that represents the integers $\mathbb{Z}$.

(4) Discuss how this construction applied to $\mathbb{Z}$ can be used to argue that models of Zermelo set theory Z contain a structure that represents the rationals $\mathbb{Q}$.

(5) If $\mathbf{T} = (T, <)$ is a linear order, we call a pair (L, R) with $L, R \subseteq T$ a *Dedekind cut* if

   (a) L is a proper initial segment of T, i.e., $L \neq T$ and if $\ell \in L$ and $t < \ell$, then $t \in L$;
   (b) R is a proper final segment of T, i.e., $R \neq T$ and if $r \in R$ and $r < t$, then $t \in R$;
   (c) R and L partition T, i.e., $R \cap L = \varnothing$ and $R \cup L = T$;
   (d) L does not have a largest element.

   We write $\mathrm{Ded}(\mathbf{T})$ for the set of Dedekind cuts of $\mathbf{T}$ and define an order on $\mathrm{Ded}(\mathbf{T})$ by $(L, R) < (L', R')$ if and only if $L \subsetneq L'$. Show that $(\mathrm{Ded}(\mathbf{T}), <)$ is a linear order.

(6) A linear order $(L, <)$ is called *complete* if every subset bounded from above has a supremum. Show that for every linear order $\mathbf{T}$, the order $(\mathrm{Ded}(\mathbf{T}), <)$ is complete.

(7) Show that the map $q \mapsto (\{x \in \mathbb{Q}\,;\, x < q\}, \{x \in \mathbb{Q}\,;\, x \geq q\})$ is an order-preserving injection from $\mathbb{Q}$ into $\mathrm{Ded}(\mathbb{Q})$ whose image lies dense in $\mathrm{Ded}(\mathbb{Q})$ and argue that models of Zermelo set theory Z contain a structure that represents the real numbers. (Do you have an idea how to define an operation of addition on $\mathrm{Ded}(\mathbb{Q})$ that extends the addition on $\mathbb{Q}$?)