Group Interaction #2

MasterMath: Set Theory
2020/21: 1st Semester
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In the second group interaction, we shall explore the nature of the recursion theorem in set theory. In class, we saw the following recursion theorem:

Recursion Theorem (on $\mathbb{N}$). Let $x_0 \in \mathbb{N}$ and $F : \mathbb{N} \to \mathbb{N}$. Then there is a unique function $G : \mathbb{N} \to \mathbb{N}$ such that

$$\begin{aligned} G(0) &= x_0 \text{ and} \\ G(n+1) &= F(G(n)). \end{aligned}$$

(1) Recapitulate the proof of the theorem and modify it to prove the following generalisation:

Recursion Theorem (from $\mathbb{N}$ into a fixed set). Let Z be any set, $z_0 \in Z$, and $F : Z \to Z$. Then there is a unique function $G : \mathbb{N} \to Z$ such that

$$\begin{aligned} G(0) &= z_0 \text{ and} \\ G(n+1) &= F(G(n)). \end{aligned}$$

(2) Let us look at other variants of the recursion theorem and check that the original proof is easily modified to yield proofs of these version. E.g.,

Recursion Theorem (Fibonacci-style). Let $x_0, x_1 \in \mathbb{N}$ and $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Then there is a unique function $G : \mathbb{N} \to \mathbb{N}$ such that

$$\begin{aligned} G(0) &= x_0, \\ G(1) &= x_1, \text{ and} \\ G(n+2) &= F(G(n), G(n+1)). \end{aligned}$$

[Here, $n+2$ abbreviates $(n+1)+1$.]

or

Recursion Theorem (nested recursions). Let $x_0, x_1 \in \mathbb{N}$ and $F_0, F_1 : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Then there are unique functions $G : \mathbb{N} \to \mathbb{N}$ and $H : \mathbb{N} \to \mathbb{N}$ such that

$$\begin{aligned} G(0) &= x_0, \\ H(0)&x_1, \\ G(n+1) &= F_0(G(n), H(n+1)), \text{ and} \\ H(n+1) &= F_1(G(n), H(n)). \end{aligned}$$

In the nested recursion theorem, check why the proof would not work anymore if the final line of the recursion equations reads "$H(n+1) = F_1(G(n+1), H(n))$". Feel free to think of other, more complicated, versions of the Recursion Theorem that are proved with the same proof idea.

(3) Go through the following alternative proof of the Recursion Theorem (using the terminology from class): Let C be the set of all relations $R \subseteq \mathbb{N} \times \mathbb{N}$ with the following properties:

(a) $(0, x_0) \in R$ and

(b) if g is a germ with $\mathrm{dom}(g) = n + 1$ and $g \subseteq R$, then $(n + 1, F(g(n)) \in R$.

(As in the lecture, a *germ* is a function whose domain is a natural number and which satisfies the recursion equations on its domain.) Show that $G := \bigcap C$ is a function that satisfies the requirements of the Recursion Theorem (in particular, $\mathrm{dom}(G) = \mathbb{N}$).