Group Interaction #14

MasterMath: Set Theory
2020/21: 1st Semester
K. P. Hart, Benedikt Löwe, Ezra Schoen, & Ned Wontner

In this group interaction we take a look at the differences between the languages of type $\mathcal{L}_{\omega,\omega}$, $\mathcal{L}_{\omega_1,\omega}$, and $\mathcal{L}_{\omega_1,\omega_1}$.
To remind ourselves:

- type $\mathcal{L}_{\omega,\omega}$ covers first-order logic:
  - countably many variables,
  - predicates and function symbols of any desired arity
  - constants
  - the usual logical connectives: $\wedge$, $\vee$, $\rightarrow$, $\neg$, ...
  - quantifiers $\forall$ and $\exists$
- type $\mathcal{L}_{\omega_1,\omega}$ extends this by having
  - $\aleph_1$ many variables
  - (countably) infinite conjunctions and disjunctions: $\bigwedge_{\eta<\alpha}\varphi_\eta$ and $\bigvee_{\eta<\alpha}\varphi_\eta$ (with $\alpha<\omega_1$)
- type $\mathcal{L}_{\omega_1,\omega_1}$ extends this even further by having
  - (countably) infinite quantifiers $\exists_{\eta<\alpha}v_\eta$ and $\forall_{\eta<\alpha}v_\eta$ (with $\alpha<\omega_1$)

We look at the languages in the context of the real line $\mathbb{R}$.
In this case the language of type $\mathcal{L}_{\omega,\omega}$ has two constants: 0 and 1, two functions: $+$ and $\times$, and a predicate (relation symbol) $<$.
We add a constant c_p for every $p\in\mathbb{R}$. We add a unary function symbol f, whose interpretation will be a function from $\mathbb{R}$ to itself, and a unary predicate A whose interpretation will be a subset of $\mathbb{R}$.

(1) Write down $\mathcal{L}_{\omega,\omega}$-sentences that express
  a. (the interpretation of) f is continuous at p.
  b. p is in the closure of (the interpretation of) A.

(2) Write down an $\mathcal{L}_{\omega,\omega}$-sentence that expresses that A is bounded from above and that it has a supremum.

(3) Let $\langle a_n : n\in\omega\rangle$ be a sequence in $\mathbb{R}$. Write down an $\mathcal{L}_{\omega_1,\omega}$-sentence that expresses that the sequence converges to π.

(4) Let X be an arbitrary countable subset of $\mathbb{R}$. Can you write down an $\mathcal{L}_{\omega,\omega}$-sentence that expresses that X is bounded from above and has a supremum? An $\mathcal{L}_{\omega_1,\omega}$-sentence? An $\mathcal{L}_{\omega_1,\omega_1}$-sentence? In each case explain why (not).

(5) The Archimedean property of $\mathbb{R}$ states: if $x,y>0$ then there is a natural number n such that $nx>y$. Can you express this in an $\mathcal{L}_{\omega,\omega}$-sentence? In an $\mathcal{L}_{\omega_1,\omega}$-sentence? In an $\mathcal{L}_{\omega_1,\omega_1}$-sentence?

(6) Write down an $\mathcal{L}_{\omega_1,\omega_1}$-sentence that expresses that f is sequentially continuous at p.

(7) Write down an $\mathcal{L}_{\omega_1,\omega_1}$-sentence that expresses that every sequence that is monotone and bounded converges.

(8) Investigate which, if any, of the statements from Exercises (3), (6) and (7) can be expressed in a weaker language.