Group Interaction #13

MasterMath: Set Theory
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Let I be an infinite set and U an ultrafilter on I. As in class we define, for functions with domain I the relations $\equiv_U$ and $\in_U$ by

$$f \equiv_U g \text{ iff } \{x \in I : f(x) = g(x)\} \in U$$

and

$$f \in_U g \text{ iff } \{x \in I : f(x) \in g(x)\} \in U$$

(1) Prove that for every function $f : I \to V$ the equivalence class $[f] = \{g : g \equiv_U f\}$ is not a set.

(2) Prove that for every function $f : I \to V$ the 'class of its elements' $\{g : g \in_U f\}$ is not a set.

We apply Scott's Trick, as in class, and redefine the equivalence class of f as

$$[f]_U = \{g : g \equiv_U f \text{ and } (\forall h)(h \equiv_U f \to \operatorname{rank} g \leqslant \operatorname{rank} h)\}$$

(3) Give an example, where $f \notin [f]_U$ (this needs an I and a U too).

(4) Prove: if $g \in_U f$ then there is an h such that $h \equiv_U g$ and $\operatorname{rank} h \leqslant \operatorname{rank} f$.

(5) Prove that for every f the entity $\{[g]_U : g \in_U f\}$ is a set.

The Mostowski collapse was used in Lecture 4 in the proof of the Theorem that every well-ordered set is isomorphic to an ordinal number. The construction was as follows: given a well-ordered set $(W, \prec)$ define $F : W \to V$ recursively by $F(x) = \{F(y) : y \prec x\}$.
In class there was an all-too-brief indication that this will work for well-founded relations too.
In our case we have specific example: the class $\mathrm{Ult}_U(V)$ and the relation $\in_U$. In the class there is no 'decreasing' $\in_U$-sequence. This is equivalent to: every subset of $\mathrm{Ult}_U(V)$ has an $\in_U$-minimal element, that is, if $X \subset \mathrm{Ult}_U(V)$ then there is an f such that $[f]_U \in X$ and there is no g such that $g \in_U f$ and $[g]_U \in X$.
Define $\pi : \mathrm{Ult}_U(V) \to V$ recursively by $\pi([f]) = \big\{\pi([g]) : [g] \in_U [f]\big\}$.

(6) Prove that π is indeed a class function into V.

(7) Prove that π is injective (the ultrapower satisfies the Axiom of Extensionality).

(8) Prove that the range, M, of π is transitive.

(9) Prove that π is an isomorphism between the structures $(\mathrm{Ult}_U(V), \in_U)$ and $(M, \in)$.

More about the Mostowski collapse can be found in Chapter 6 of Jech, pages 67–69.