## Group Interaction \#12

MasterMath: Set Theory

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This group interaction is devoted to an introduction to ultrapowers. If you have not seen ultrapowers before then make sure you work problems (1) through (6) thoroughly.
Let $X$ and $I$ be sets and let $U$ be an ultrafilter on $I$. Define a relation $\equiv\left(\right.$ sometimes $\left.\equiv_{U}\right)$ on the set $X^{I}$ by

$$
f \equiv g \text { iff }\{i: f(i)=g(i)\} \in U
$$

(1) Prove that $\equiv$ is an equivalence relation.

We let $[f]$ (sometimes $[f]_{U}$ ) denote the equivalence class of $f$. The set $X^{I} / \equiv$ of equivalence classes is the ultrapower of $X$ by the ultrafilter $U$.
For $x \in X$ let $x: I \rightarrow X$ be the constant function with value $x$.
(2) Prove: if $X$ is finite then the ultrapower is equal to $\{[\boldsymbol{x}]: x \in X\}$.
(3) Prove: if $U$ is a principal ultrafilter then the ultrapower is equal to $\{[\boldsymbol{x}]: x \in X\}$.
(4) Prove: if $U$ is nonprincipal and $|I| \leqslant|X|$ then the ultrapower is not equal to $\{[\boldsymbol{x}]: x \in X\}$.

Define a relation $\in^{*}$ on the ultrapower of $X^{I} / \equiv$ by

$$
[f] \in^{*}[g] \text { iff }\{n: f(n) \in g(n)\} \in U
$$

(5) Prove that this definition is independent of the choice of representatives: if $f \equiv f^{\prime}$ and $g \equiv g^{\prime}$ then $[f] \in^{*}[g]$ iff $\left[f^{\prime}\right] \in^{*}\left[g^{\prime}\right]$.
The set $X^{I} / \equiv$ together with the relation $\in^{*}$ is called the ultrapower of the structure $(X, \in)$ by $U$.
Using the constant functions $\boldsymbol{x}$, for $x \in X$ we define a map $j$ (or $j_{U}$ ) from $X$ into $X^{I} / \equiv$ by $j: x \mapsto[\boldsymbol{x}]$.
(6) Prove that if $U$ is principal or if $X$ is finite then $j_{U}$ is an isomorphism.

It is a consequence of a general theorem of Loś that $j$ is an elementary embedding of structures and that $(X, \in)$ and $\left(X^{I} / \equiv, \epsilon^{*}\right)$ are elementarily equivalent.
This general result has as its basis the following statement (formulated for the present situation): if $\varphi\left(x_{1}, \ldots, x_{k}\right)$ is a formula with its free variables shown then for every $k$ many members $f_{1}, \ldots, f_{k}$ of $X^{I}$

$$
\varphi\left(\left[f_{1}\right], \ldots\left[f_{k}\right]\right) \text { holds in }\left(X^{I} / \equiv, \in^{*}\right)
$$

if and only if the set of $n$ such that

$$
\varphi\left(f_{1}(n), \ldots, f_{k}(n)\right) \text { holds in }(X, \in)
$$

belongs to $U$.
A proof of this statement and the Theorem of Łoś can be found in the books of Mendelson (Introduction to Mathematical Logic, pages 129-135 of the sixth edition) and Van Dalen (Logic and Structure, section 4.5 of the fifth edition) for general languages, and also in Chapter 12 of Jech's book, for the language of set theory.

As an example of the kind of ultrapower that we shall be dealing with we now take $X=V_{\omega}, I=\omega$, and we let $U$ be a non-principal ultrafilter on $\omega$.
The theorem of Łoś implies that in this case the ultrapower is a model of FST.
This can also be verified directly for each individual axiom; here are some examples.
(7) Given $f, g \in V_{\omega}^{\omega}$ define $h(n)=\{f(n), g(n)\}$. Prove: $[f] \in^{*}[h],[g] \in^{*}[h]$, and: if $[k] \in^{*}[h]$ then $k \equiv f$ or $k \equiv g$.
Thus we see that the Pairing Axiom holds in the ultrapower.
(8) Show that the Axiom of Regularity holds in the ultrapower: given $f$ such that $[f] \neq[\emptyset]$ construct $g$ such that $[g] \in^{*}[f]$ and there is no $h$ such that $[h] \in^{*}[g]$ and $h \in^{*}[f]$.
The image of $V_{\omega}$ is an initial segment of the ultrapower:
(9) Prove: if $x \in V_{\omega}$ and if $[f] \in^{*}[\boldsymbol{x}]$ then there is a $y \in x$ such that $f \equiv \boldsymbol{y}$.

The ultrapower is much larger than the structure $V_{\omega}$ itself.
Let $d: \omega \rightarrow \omega$ be the identity function.
(10) Prove that $[\boldsymbol{n}] \in^{*}[d]$ for all $n \in \omega$ and: if $x \in V_{\omega}$ and $[\boldsymbol{x}] \in^{*}[d]$ then $x \in \omega$.
(11) Give an example of an $f: \omega \rightarrow \omega$ such that $[f] \in^{*}[d]$, and $[\boldsymbol{n}] \in^{*}[f]$ for all $n \in \omega$.
(12) Improve the previous part by constructing a sequence of functions $\left\langle f_{m}: m \in \omega\right\rangle$ such that $f_{0}=d$, $\left[f_{m+1}\right] \in^{*}\left[f_{m}\right]$ for all $m$, and $[\boldsymbol{n}] \in^{*}\left[f_{m}\right]$ for all $n$ and $m$. In addition construct a function $g$ such that $[\boldsymbol{n}] \in^{*}[g] \in\left[f_{m}\right]$ for all $n$ and $m$.
(13) Why does the previous part not contradict the Axiom of Regularity?
(14) The basic statement can be proved by induction on the complexity of the formula; you can give this a try if you are familiar with this kind of argument.

