Group Interaction #11

MasterMath: Set Theory
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This group interaction wants you to reflect on some of the steps in the proofs of the Erdős-Dushnik-Miller theorem and Kőnig's Infinity Lemma.

(1) In the proof of the E-D-M theorem the following statement was left unproven and deferred to this group interaction:

> Let $F : [X]^n \to k$ be a colouring. Then every homogeneous set for F can be extended to a *maximal* homogeneous set.

Prove this statement.

(2) The proof of the E-D-M theorem in class was split into two cases: κ regular and κ singular.
This exercise shows that the proof for the singular case also works in the regular case.
To see this let $F : [\kappa]^2 \to 2$ be a colouring; in the proof we considered two cases:

> (1) every subset, X, of κ of cardinality κ has an element, x, such that $|B(x) \cap X| = \kappa$, and
>
> (2) there is a subset, X, of κ of cardinality κ such that for every $x \in X$ we have $|B(x) \cap X| < \kappa$

We proved that in case (1) there is an infinite Blue homogeneous set.
We proved that in case (2) there is a Red homogeneous set of cardinality κ, for a singular κ.

Prove that in case (2), with κ regular, there is a Red homogeneous set of cardinality κ.
Compare the length of your proof with that for the singular case.

(3) In the proof of Kőnig's Lemma we constructed an infinite branch by recursion. The formulation was somewhat informal and seems to involve a choice:

> given $x_n \in T_n$ such that $\{y : y > x_n\}$ is infinite take $x_{n+1} \in T_{n+1}$ above x_n such that $\{y : y > x_{n+1}\}$ is infinite.

a. Identify what kind of function is needed in order to apply the Recursion Principle to construct the branch.
b. Prove the existence of such a function by an application of the Axiom of Choice to a suitable family of sets.
c. Alternatively: show how to *define* such a function from a free ultrafilter on T.