Group Interaction #10

MasterMath: Set Theory
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K. P. Hart, Benedikt Löwe, Ezra Schoen, & Ned Wontner

This group interaction wants you to investigate further some aspects of the proofs of Ramsey's theorem and of the Ramsey property of selective ultrafilters.

The first proof used, both in the case $n = 2$ as in the general inductive step a free ultrafilter, U, to guide the recusive construction of a sequence $\langle a_l : l \in \omega \rangle$: the given function $F : [\omega]^{n+1} \to k$ was used to specify for every $x \in [\omega]^n$ a set A_{x,i_x} in U on which the auxiliary function $b \mapsto F(x \cup \{b\})$ was constant. At each stage a_l was chosen to be the minimum of the intersection of the sets A_{x,i_x} for $x \in [\{a_i : i < l\}]^n$. As a consequence: if we let $K = \{a_l : l \in \omega\}$ then the restriction of F to $[K]^{n+1}$ is such that for every $x \in [K]^{n+1}$ the value $F(x)$ depends only on its first n elements.

The second proof used the inductive assumption repeatedly to find, given $F : [\omega]^{n+1} \to k$, for every $m \in \omega$ a set $H_m \in U$ such that the function $z \mapsto F(\{m\} \cup z)$ was constant on $[H_m]^n$, with value i_m, say. The sequence $\langle x_m : m \in \omega \rangle$ that resulted from an application of selectivity had the property that the value of $F(z)$ for $z \in [\{x_m : m \in \omega\}]^{n+1}$ would depend only on its minimum: $F(z) = i_p$, where $p = \min z$.

(1) Modify the second proof so that it yields a proof of Ramsey's theorem. *Hint*: You can make H_{m+1} an infinite subset of H_m.

(2) Modify the first proof so that it does not mention ultrafilters; that is, show that one can specify the values i_x in such a way that the intersection of the sets A_{x,i_x} is infinite.

(3) Can you modify the first proof so that it yields a proof of the Ramsey property of selective ultrafilters?