Group Interaction #1

MasterMath: Set Theory
2020/21: 1st Semester
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(1) Every week, there will be one *group interaction* of 45–50 minutes. The group interactions take place remotely via Zoom.

(2) A group interaction consists of two to three students who work together on a work sheet in the presence of one of the two teaching assistants (Ezra Schoen or Ned Wontner). A group does not have to cover the entire work sheet.

(3) Students are expected to actively participate in these group interaction sessions each week. The *group interaction score* is the number of times a student actively participated in one of the group interaction sessions (the maximum score is **10**). [Students who are sitting the *Rudiments* exam are only expected to participate in weeks one to seven, but are welcome to continue to participate. Their score is multiplied by two (with a maximum score of **10**).

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In the first group interaction, we shall construct graph models of weak set theories. These constructions happen in naïve set theory and you are allowed to use all of the tools of ordinary mathematics (i.e., recursion and induction) to prove things about your models.

Let $\mathbf{G} = (V, E)$ be any graph. If $v \in V$, we write $\mathrm{pred}_{\mathbf{G}} v := \{w \in V\,;\, w\ E\ v\}$ for the set of $\mathbf{G}$-predecessors of v. We write $[V]^{\leq 2}$ for the set of all subsets of V of at most two elements. If $Z \in [V]^{\leq 2}$, we say that Z is *covered in* $\mathbf{G}$ if there is some $v \in V$ such that $\mathrm{pred}(v) = Z$. Otherwise, we say that Z is *uncovered in* $\mathbf{G}$.

The graph $\mathrm{p}(\mathbf{G}) := (V^*, E^*)$ is called the *pairing augmentation of* $\mathbf{G}$ if V^* consists of all of the vertices of V plus a set of new vertices V^+ such that each new vertex $v \in V^+$ corresponds to exactly one set $Z \in [V]^{\leq 2}$ that is uncovered in $\mathbf{G}$ with $\mathrm{pred}_{\mathrm{p}(\mathbf{G})}(v) = Z$. Furthermore, for each $v \in V$, $\mathrm{pred}_{\mathbf{G}}(v) = \mathrm{pred}_{\mathrm{p}(\mathbf{G})}(v)$.

Given any graph $\mathbf{G}$, we define by recursion

$$\mathbf{G}_0 := \mathbf{G} \text{ and}$$
$$\mathbf{G}_{n+1} := \mathrm{p}(\mathbf{G}_n).$$

Write $\mathbf{G}_n := (V_n, E_n)$ and define $V_\infty := \bigcup_{n\in\mathbb{N}} V_n$ and $E_\infty := \bigcup_{n\in\mathbb{N}} E_n$. We call the graph $\mathbf{G}_\infty := (V_\infty, E_\infty)$ the *pairing closure of* $\mathbf{G}$.

(i) Show that for every graph $\mathbf{G}$, $\mathbf{G}_\infty$ is a model of the pairing axiom.

(ii) Show that if $\mathbf{G}$ was extensional, then so is $\mathbf{G}_\infty$.

Now, let us start with $\mathbf{G}^\bullet = (\{\bullet\}, \varnothing)$, the graph with a single vertex and no edges and consider its pairing closure $\mathbf{G}^\bullet_\infty$.

(iii) Get a feeling for the construction by drawing $\mathbf{G}^\bullet_1$, $\mathbf{G}^\bullet_2$, $\mathbf{G}^\bullet_3$, and $\mathbf{G}^\bullet_4$.

(iv) Check the validity of the union axiom and the power set axiom in $\mathbf{G}^\bullet_\infty$. State your finding as an independence result for axioms of set theory.

(v) Show that $\mathbf{G}^\bullet_\infty$ satisfies the axiom scheme of separation. [*Hint.* Show first that for every vertex $v \in V^\bullet_\infty$, we have that $\mathrm{pred}(v)$ has at most two elements. This helps us to reduce the claim of the axiom scheme of separation to something more manageable.]

We can now play around with pairing closures by changing the graph we are starting with. Can you construct a model of pairing that is *not* a model of extensionality this way? Can you construct a model of pairing that is *not* a model of the axiom scheme of separation?

In order to get models that satisfy the power set axiom, you can modify the above construction: again, let $\mathbf{G} = (V, E)$ be an arbitrary graph. For each $v \in V$, we can consider $P(v) := \{w \in V\,;\, w \text{ is a } \mathbf{G}\text{-subset of } v\} \subseteq V$. We call a vertex v *handled in* $\mathbf{G}$ if there is some $w \in V$ such that $\mathrm{pred} w = P(v)$. Otherwise, we say that v is *unhandled in* $\mathbf{G}$. The *power set augmentation of* $\mathbf{G}$ is defined in the same way as the pairing augmentation, but the new vertices in V^+ have as predecessors precisely the sets $P(v)$ for a vertex v that was unhandled in $\mathbf{G}$. We write $\mathrm{pow}(\mathbf{G})$ for the power set augmentation of $\mathbf{G}$ and define the *power set closure* in the same way.

(vi) Show that if $\mathbf{G}$ was extensional, then the power set closure of $\mathbf{G}$ is extensional.

(vii) Show that if $\mathbf{G}$ was locally finite, then the power set closure of $\mathbf{G}$ satisfies the power set axiom. (What if $\mathbf{G}$ is not locally finite?)

(viii) Start with $\mathbf{G}^\bullet$ and consider its power set closure. Again, start with drawing the graphs of the first four or five steps of the recursion to get a feeling for the construction.

(ix) Check the validity of the pairing axiom, the union axiom, and the axiom scheme of separation in the power set closure of $\mathbf{G}^\bullet$.

If there is any time left, you can think about whether the techniques given on this sheet will allow you to construct a model of extensionality, pairing, power set, but not union.